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ASYMMETRY DEFINITION OF A CONVEX BODY BY MEANS OF CHARACTERISTIC POINTS

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ABSTRACT

Different approaches for centering estimation of a point x in a planar convex body are proposed. They lead to measures of symmetry and critical points. This paper recalls that symmetry estimation is deeply related to underlying metrics.

Keywords : shape parameters, metrics, asymmetry, convex body.

INTRODUCTION

In the field of Image Analysis the lack of mathematical tools often forbids rigorous developments. On the other hand, the set of convex bodies has been thoroughly studied providing a lot of concepts and techniques whose interest for applications is undoubtful.

The aim of this paper is to bring some answers to the question : "How to define the centering degree of a point belonging to a convex body?" In order to give a sound foundation to such concepts, we need to introduce different real valued functions for centering estimation of a point x in a convex body K. For each of them, the critical point x* realizing the best centering leads to a measure of symmetry.

Let us recall (Grünbaum, 1963) that a real valued function F defined on the set $\mathcal{K}$ of planar convex bodies (i.e. the set of compact convex sets in IR^2 with non empty interior) is a measure of symmetry provided :

For every convex body K $\quad 0 \leq F(K) \leq 1$

$F(K)=1 \Leftrightarrow$ K is centrally symmetric

$F(K) = F(T(K))$ for every affine transformation T.

F is continuous (on the set of classes of convex bodies equivalent under affine transformations).

Symmetry parameters and metrics are deeply linked. From a metric d on $\mathcal{K}$, a measure of symmetry can be deduced by $F(K) = \exp(- d_1(K, \mathbb{S}))$ where $\mathbb{S}$ denotes the subset of $\mathcal{K}$ consisting of centrally symmetric convex bodies (symmetric with respect to an interior point) and $d_1(K, \mathbb{S}) = \inf \{d(K, S) \,;\, S \in \mathbb{S}\}$.

BESICOVITCH FUNCTION

Let x be a point in K and $K^-(x)$ the symmetrical set of K with respect to x. Let $\mu(K)$ design the area of K (see Fig. 1).

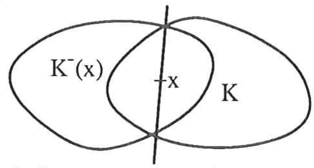

Fig. 1. Representation of K and $K^-(x)$

The centering of x (Besicovitch, 1948) is estimated by :

$$f_1(x, K) = \frac{\mu(K \cap K^-(x))}{\mu(K)} \qquad (1)$$

which gives the proportion filled by the maximal central convex body included in K, centered at x. The upper bound of $f_1(x, K)$ is reached at a unique point x_1^* in K, called Besicovitch critical point. which is a consequence of the strict convexity and compactness of the sets $\{x \in K ; f_1(x, K) \geq a\}$ (for $a > 0$) (see the proof in Stein, 1956)

x_1^* is the center of the biggest central convex body included in K.

Let $F_1(K) = f_1(x_1^*, K)$, the following properties hold :

$\forall K \in \mathcal{K}$ $\quad F_1(K) \in [2/3, 1]$ (Besicovitch, 1948)

$F_1(K) = 2/3$ $\quad \Leftrightarrow$ K is a triangle (Fary, 1950)

$F_1(K) = 1$ $\quad \Leftrightarrow$ K is centrally symmetric

It' s easy to prove that **$F_1(K)$ is a measure of symmetry of K.**

The Besicovitch function.can be linked with the metric d_Δ defined by :

$$d_\Delta(A, B) = \mu(A \cup B) - \mu(A \cap B) \qquad \forall (A, B) \in \mathcal{K}^2.$$

which can also be expressed : $\quad d_\Delta(K, K^-(x)) = 2\,\mu(K)\,(1 - f_1(x, K))$

thus the following result holds : $\quad \mathbf{f_1(x, K) = 1 - \frac{d_\Delta(K, K^-(x))}{2\mu(K)}} \qquad (2)$

WINTERNITZ FUNCTION (Grünbaum, 1963)

For each point x of K, and each oriented line D_θ through x, we consider the ratio of the areas of the two parts of K (left and right) determined by D_θ (see Fig. 2).

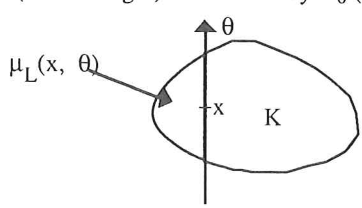

Fig. 2 . partition of the area of K into $\mu_L(x, \theta)$ and $\mu_R(x, \theta)$

For Winternitz the centering of x is estimated by the lowest ratio of these two areas when $\theta \in [0, 2\pi]$. Thus the Winternitz function can be defined by :

$$f_2(x, K) = \text{Inf}\ \{\frac{\mu_L(x, \theta)}{\mu_R(x, \theta)} / \theta \in [0, 2\pi]\} \quad (3)$$

The same argument as in the precedent paragraph proves that the upper bound of $f_2(x, K)$ is reached at a unique point x_2^* in K called Winternitz critical point.

Thus

$$f_2(x_2^*, K) = \text{Sup}\ \{ f_2(x, K) / x \in K \} \quad (4)$$

Let $F_2(K) = f_2(x_2^*, K)$, the following properties hold (Eggleston, 1958 ; Grünbaum, 1963)

$\forall K \in \mathcal{K}$ $\quad F_2(K) \in [4/5, 1]$

$F_2(K) = 4/5$ $\quad \Leftrightarrow$ K is a triangle.

$F_2(K) = 1$ $\quad \Leftrightarrow$ K is centrally symmetric.

$F_2(K)$ is a measure of symmetry of K.

Until now we did not find a metric which could be linked with F_2.

MINKOWSKI FUNCTION (Grünbaum, 1963)

Let $x \in K$ and $\theta \in [0, 2\pi]$. $h_K(x, \theta)$ (respectively $h_K(x, \theta+\pi)$) is the euclidean distance between x and the support line D_θ (respectively D_θ) of direction $\theta+\pi/2$ (respectively $\theta+3\pi/2$) (see Fig. 3a).

The following Minkowski function f_3 estimates the centering of x related to the support lines D_θ and $D_{\theta+\pi}$ where $\theta \in [0, 2\pi]$, i.e. the lower bound of the mapping $\theta \dashrightarrow \frac{h_K(x, \theta)}{h_K(x, \theta+\pi)}$

$$f_3(x, K) = \text{Inf}\ \{ \frac{h_K(x, \theta)}{h_K(x, \theta+\pi)} / \theta \in [0, 2\pi] \} \quad (5)$$

As in the precedent paragraphs, one can prove (Grünbaum, 1963) that there exists also a unique critical point x_3^* called Minkowski critical point which realizes the upper bound of $f_3(x, K)$ when x moves in K.

$$f_3(x_3^*, K) = \text{Sup}\ \{f_3(x, K) / x \in K \} \quad (6)$$

Let $F_3(K) = f_3(x_3^*, K)$, the following properties hold (Grünbaum, 1963) :

$\forall K \in \mathcal{K}$ $\quad F_3(K) \in [1/2, 1]$

$F_3(K) = 1/2$ $\quad \Leftrightarrow$ K is a triangle.

$F_3(K) = 1$ $\quad \Leftrightarrow$ K is centrally symmetric.

$F_3(K)$ is a measure of symmetry of K.

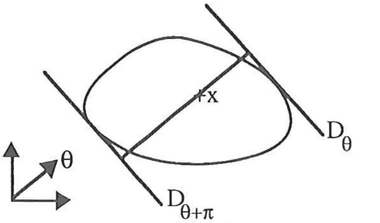

Fig. 3a

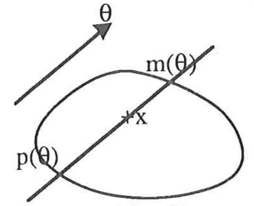

Fig. 3b

Support lines of direction θ and a chord of direction θ through x

Let [m(θ), p(θ)] be a chord in K of direction θ through the point x (see Fig. 3b).

Let $$g(x, K) = \mathrm{Inf}\{ \frac{xm(\theta)}{xp(\theta)}, \theta \in [0, 2\pi] \} \quad (7)$$

where xm(θ) and xp(θ) denote the lengths of the segments [xm(θ)] and [xp(θ)].
We can prove that

$$\mathbf{g(x, K) = f_3(x, K)} \quad (8)$$

That result is a consequence of the inclusions :

$$K \supset H(x, -f_3(x, K))(K) \text{ and } K \supset H(x, -g(x, K))(K)$$

(where H(x, λ)(K) denotes the homothetic to K centered at x in the ratio λ) which implies :

$$\forall \theta \in [0, 2\pi] \quad xm(\theta) > f_3(x, K)\ xp(\theta) \text{ and } h_K(x, \theta) > g(x, K)\ h_K(x, \theta+\pi))$$

It means the best centering for Minkowski is the same if we consider the distances on the chords of K (radial functions) or the distances to parallel support lines of K (support functions).

Let us note that Hammer (1951) studied the infimum g(x, K) ; he did not notice that it was the Minkowski ratio but he proved that if the centroid of K is the point of trisection of any chord then K is a triangle.

f_3 can be linked with Asplund distance:
Let us recall that the Asplund distance $d_A(K, L)$ (K and L elements of $\mathcal{K}$) is defined by :

$$d_A(K, L) = \mathrm{Log}\ (\inf \{ \frac{\alpha}{\beta};\ (\alpha, \beta) \in R^{+*2} ; \alpha K \supset L \text{ and } L \supset \beta K \text{ up to a translation}\})$$

The following result holds :

$$f_3(x_3{}^*, K) = \exp(-d_A(K, S(K))$$

where S(K) denotes the Minkowki symmetrical set of K :

$$S(K) = 1/2(K \oplus K^-) \quad \text{(Delfiner,1979)}$$

where K^- is the symmetrical set of K with respect to the origin and ⊕ is the Minkowski addition defined, for A and C two compact sets of IR^2, by $A \oplus C = \{a + c ; a \in A, c \in C\}$.
Since $d_A(K, S(K)) = d_A(K, \mathbb{S})$ it follows

$$\mathbf{f_3(x_3{}^*, K) = \exp(-d_A(K, \mathbb{S}))}.$$

For the precedent measures, the triangle is the most asymmetric convex body.
If K is a triangle its centroid is at the same time the Winternitz, Besicovitch and Minkowski critical point. It is not true in general.

OTHER CHARACTERISTIC POINTS

The use of chords in a convex body leads to other characteristic points (see Fig. 4).

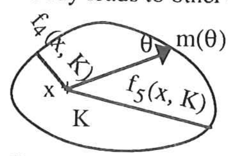

Fig. 4. visualisation of $f_4(x, K)$ and $f_5(x, K)$

Let $f_4(x, K) = \text{Inf}\ \{xm(\theta) / \theta \in [0, 2\pi]\}$ and $f_5(x, K) = \text{Sup}\ \{xm(\theta) / \theta \in [0, 2\pi]\}$

$f_4(x, K)$ is the euclidean distance from x to the boundary of K. The radius r(K) of the largest disk inscribed in K can be expressed by :

$$r(K) = \text{Sup}\ \{f_4(x, K) / x \in K\} \qquad (9)$$

This upperbound is reached by at least one point x_4^* (incentre) belonging to the ultimate morphological eroded set of K.

$f_5(x, K)$ is the greatest distance from x to a point of the edge of K. The radius R(K) of the smallest disk circumscribed to K can be expressed by :

$$R(K) = \text{Inf}\ \{f_5(x, K) / x \in K\} \qquad (10)$$

This lower bound is reached by a unique point x_5^* (circumcentre).

x_4^* and x_5^* are characteristic points of K because they are similarity invariant (i.e. invariant under similarities). They do not lead to asymmetry measures but to a well known circularity measure

$C(K) = \dfrac{r(K)}{R(K)}$ which realizes :

$\forall K \in \mathcal{K} \quad C(K) \in]0, 1]$

$C(K) = 1 \Leftrightarrow$ K is a disk.

$C(K) = \exp(-d_A(K, B))$ where B is the (unit) disk.

SOME EXAMPLES

Let K_1 and K_2 be two convex bodies (see Fig. 5). In these examples, the characteristic points are placed on the symmetry line (oX).We have also considered the situation (on the line (oX)) of the centroid (denoted by g) of each body, because it seems interesting to compare the proximity of the centroid with all the characteristic points. Their respective abscisses and associated symmetry measures are presented in table 1. It permits to conclude that K_2 is less symmetric than K_1 with respect to all these symmetry measures.

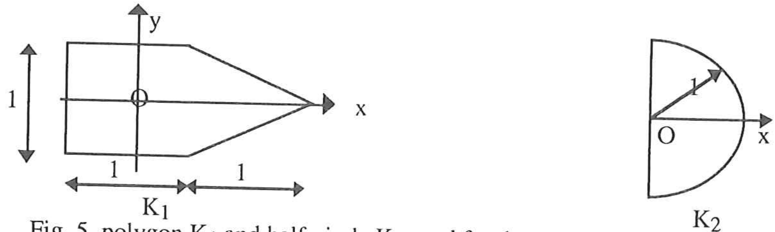

Fig. 5. polygon K_1 and half-circle K_2 used for the numerical application

Table 1 . values of the different symmetry coefficients for the shapes K_1 and K_2

	Xg	X1*	F1(K)	X2*	F2(K)	X3*	F3(K)
K1	5/18	0.25	5/6	$1-1/\sqrt{2}$	≈ 0.892	1/3	5/7
	≈ 0.277		≈0.833	≈ 0.293		≈ 0.333	≈ 0.714
K2	$4/3\pi$	≈0.447	≈ 0.819	≈ 0.430	≈ 0.885	$\sqrt{2}-1$	$2-\sqrt{2}$
	≈0.424					≈ 0.414	≈ 0.586

The determination of the critical points x_i* is not easy, in general.
We can evaluate the centering of the centroid g of K by using $f_i(g, K)$ for i = 1, 2, 3.
As g is an affine invariant point $f_i(g, K)$ is a measure of symmetry of K. It has been proved that the bounds of $f_i(g, K)$ are the same as $F_i(K)$ (Grünbaum, 1963).
We give some results for the precedent examples :

$f_1(g, K_1) \approx 0.831$ $f_2(g, K_1) \approx 0.876$ $f_3(g, K_1) \approx 0.636$.

$f_1(g, K_2) \approx 0.815$ $f_2(g, K_2) \approx 0.871$ $f_3(g, K_2) \approx 0.695$.

Thus the centroid of K_1 is less centered in K_1 than the centroid of K_2.

CONCLUSION

We have tried in this paper to give again some interest for symmetry measures discovered but not explored by a few mathematicians of the beginning and the middle of the century. Such measures were not implemented until now. The improvement of informatical tools allows to compute the critical points and their associated symmetry measures: see for example Moreau (1987) and Rubio (1990) for the computation of Besicovitch and Minkowski coefficients. After noticing the nearness of the centroid with the critical points, the algorithms start from the centroid and detect the "best" neighbouring point (for the considered function f_i). The properties of continuity of these measures are useful when we have to evaluate a distortion of a planar shape with a reference one.

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