Bulletin de la Société Royale des Sciences de Liège, Vol. 73, 1, 2004, pp. 7-19

BARRELLEDNESS AND DUAL STRONG SEQUENCES IN LOCALLY CONVEX SPACES

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Abstract

Given a dual pair $\langle E, F \rangle$, we consider a series of bidual enlargements of F, define new concepts of dual strong sequence and dual strong union and analyze their connections to barrelledness.

Mathematics subject classification : 46A08
Key-words : Locally barrelled, bidual, quasi-distinguished, Banach-Mackey property

1 Introduction : nonbarrelled locally convex spaces

Given a locally convex space E, we introduce the notions of a dual strong sequence and dual strong union. The actualization of these notions is motivated by exploring sets with "nice" useful properties such as compactness or equivalence of different topologies, ([8], [10], [26], [27]). Along with importance and value in mathematics, sets with "nice" properties are essential in applicative areas, such as economics, engineering, physics, where general infinite-dimensional locally convex spaces are widely used. Examples of "nice" properties employed in applications can be found in [1] or [20]. In the forthcoming article we show that dual strong sequences are a natural resource for some of the "nice" properties. In the present article we investigate the relationship between the dual strong sequences and barrelledness conditions.

A dual strong sequence is a universal structure in the sense that it appears in any locally convex nonbarrelled space. The collection of nonbarrelled spaces is considerable. Apparently the most important are the spaces of continuous functions $C(T)$, widely used in different applications. It is known that the space $C_p(T)$ of real continuous functions on a completely regular Hausdorff space T is barrelled for the topology of pointwise convergence if and only if each bounding subset of T is finite, ([6]), which implies that for a barrelled space $C(T)$ the topologies of uniform convergence on finite, compact and bounding sets of T coincide. Thus if T admits an infinite bounding (or compact) subset then $C_p(T)$ is nonbarrelled. The generalizations of Buck's strict topologies on $C(T)$ explored by Ruess are neither barrelled nor even σ-evaluable, however they enjoy the Banach-Mackey property, ([27], Proposition 2.1, σ-evaluable = ω-quasi-barrelled). Emphasizing their importance, we notice that each of the strict topologies of [27] is localizable on any absorbing disk (i.e. it is the finest locally convex topology equivalent to the given one, see also [26]). We also point out that in the spaces of continuous vector valued functions there exist strict topologies with important properties of angelicity and metrizability of compact sets, ([8], Corollary 1.11). Regarding the spaces of vector valued functions, we mention the result of [16] that the space $c_0(\Omega, X)$ with the sup norm is barrelled if and only if X is barrelled. Notice that $c_0(\Omega, X)$ can be regarded as the generalization of the well-known space $c_0(\mathbb{N}, X) = c_0(X)$ of convergent to zero sequences in X (see [17], t. I, p. 463). As to general locally convex spaces, every separable Banach space admits a finer nonbarrelled Mackey topology of sup type, ([12], Example 3.5). Every Banach space admits a strictly increasing sequence of nonbarrelled norms, ([21], Proposition 6(a)). The projective tensor product $\mathbb{K}^{\mathbb{N}} \otimes_\pi \mathbb{K}^{(\mathbb{N})}$ is nonbarrelled, as well as $E \otimes_\pi \mathbb{K}^{(\mathbb{N})}$ for a metrizable non-normable space E, [5]. According to Lurje, the ℓ_p-direct sum $(\oplus_{n\in\mathbb{N}} X_n)_p$ of normed spaces X_n is barrelled if and only if each X_n is barrelled, ([23], 4.9.17). Since any necessary and sufficient requirement for being barrelled provides as well a condition for being nonbarrelled, we suggest [23] as a valuable resource for further nonbarrelled examples.

Working in the general frame of nonbarrelled locally convex Hausdorff spaces we develop a broad generic approach, presenting a fairly complete account of concise results and leaving the study of tangible spaces to further investigations.

2 Preliminary notations and definitions

We follow the definitions and notations of [17] or [22]. Let E be an infinite-dimensional vector space over the field $\mathbb{K}$ of the real or complex numbers, E^* the algebraic dual of E and F a $\sigma(E^*, E)$-dense subspace of E^*. Given a dual pair $\langle E, F\rangle$, we denote by $\beta(E, F), \mu(E, F), \sigma(E, F)$ the strong, Mackey and weak topologies on E, respectively. We denote by $\beta^*(E, F)$ the topology on E of uniform convergence on all strongly bounded sets of F. All topologies on E will be considered locally convex and Hausdorff.

A *disk* is an absolutely convex set. A singleton $\{ax : a \in \mathbb{K}, |a| \leq 1\}$ for some nonzero $x \in E$ is a disk as well as E itself. The set $\{0\}$ is a trivial disk. All coming next disks will be nontrivial. A disk is *absorbing* (*bornivorous*), if it absorbs any element (bounded set) of E. A *barrel* is a closed absorbing disk. For a disk A we denote by E_A the linear hull of A, equipped with its gauge g_A. The disk A is *barrelled*, resp. *Banach*, if E_A is barrelled, resp. Banach. It is *finite-dimensional*, resp. *infinite-dimensional*, if E_A is finite-dimensional, resp. infinite-dimensional.

A *family* $\mathbf{A}$ of disks in E is a set of disks covering E such that for any A, $B \in \mathbf{A}$ and α, $\beta \in \mathbb{K}$ there is $C \in \mathbf{A}$ satisfying : $\alpha A + \beta B \subseteq C$. We consider the following families :

(1) **B** – the family of all $\sigma(E, F)$-closed bounded disks of E.

(2) **B*** – the family of all $\sigma(E, F)$-closed strongly bounded disks of E.

(3) **NBAR** – the family of all $\sigma(E, F)$-closed bounded barrelled disks of E.

(4) **BAN** – the family of all $\sigma(E, F)$-closed bounded Banach disks of E.

(5) **WC** – the family of all weakly compact disks of E.

(6) **FIN** – the family of all finite-dimensional disks of E.

The name of types 3, 4, 5, 6 is a semantic abbreviation of the property the family carries, for example, **NBAR** = Normed BARrelled. Notice that a family of type $(n+1)$ is embedded into the family of type (n) for $n = 1, 2, 3, 4, 5$.

There are special notions for a locally convex dual pair $\langle E, F\rangle$ with some of the families equal. We say that E is *Banach-Mackey* if $\mathbf{B} = \mathbf{B}^*$ in E. Since barrels and closed bounded disks of a dual pair $\langle E, F\rangle$ are connected by polarity (the polar of a barrel is a closed bounded disk and vice versa) we observe that E is *Banach-Mackey* if and only if F is *Banach-Mackey*. We say that E is *locally barrelled*, resp. *locally complete* (or *ℓ^q-complete*, see [24]) if $\mathbf{B} = \mathbf{NBAR}$, resp. $\mathbf{B} = \mathbf{BAN}$ in E. We say that E is *locally quasi-barrelled*, resp. *locally quasi-complete*, if $\mathbf{B}^* = \mathbf{NBAR}$, resp. $\mathbf{B}^* = \mathbf{BAN}$, in E. The space E is *dual locally (quasi) barrelled*, resp. *dual locally (quasi) complete*, resp. *(quasi) barrelled* for $\mu(E, F)$ if $\mathbf{B} = \mathbf{NBAR}$ ($\mathbf{B}^* = \mathbf{NBAR}$), resp. $\mathbf{B} = \mathbf{BAN}$ ($\mathbf{B}^* = \mathbf{BAN}$), resp. $\mathbf{B} = \mathbf{WC}$ ($\mathbf{B}^* = \mathbf{WC}$), in F. Dual locally (quasi) complete and (quasi) barrelled spaces are standing next to each other, which is not a coincidence, because we know that these properties are affiliated, ([24], [28]). Later we prove (Proposition 5.2) that for a dual strong union the case of $\mathbf{B} = \mathbf{NBAR}$ in F and the barrelledness of E are connected as well. If $\mathbf{B} = \mathbf{FIN}$ in E, then $F = E^*$ and $\sigma(E, F) = \beta(E, F)$ is the finest locally convex topology on E. If $\mathbf{B} \neq \mathbf{FIN}$ in E, then F admits an enlargement in E^* and E admits a finer locally convex topology. Different enlargements of F (including countable enlargements) and their connection to the existence of infinite-dimensional bounded sets in E were studied in ([12], [29],[30]). We also proved in [32] that any locally convex space of dimension at most c with an infinite-dimensional bounded Banach disk admits a hyperplane satisfying $\mathbf{B} \neq \mathbf{BAN} = \mathbf{FIN}$. Although the **NBAR** family is less nurtured, any dense countable-codimensional subspace in a Banach space is an example of a space with a nonempty **NBAR** family such that $\mathbf{B} = \mathbf{NBAR} \neq \mathbf{BAN}$. Since a locally complete metrizable space is complete (see [30], p. 7), it follows that a dense countable codimensional subspace of a Frechet space provides another example with $\mathbf{B} = \mathbf{NBAR} \neq \mathbf{BAN}$. Selecting a locally complete space with an infinite dimensional bounded set of dimension at most c and applying the method of [32] we obtain a hyperplane satisfying $\mathbf{FIN} = \mathbf{BAN} \neq \mathbf{NBAR} = \mathbf{B}$. We also mention the result of Valdivia on an increasing sequence of subspaces in a specific Banach space with at least one (therefore all but a finite number) of the subspaces dense and barrelled, ([36], Theorem 1, p. 46). The result of [36] is related to the measure theory thus incorporating the spaces with $\mathbf{B} = \mathbf{NBAR}$ into the eminent coalition of applicative spaces.

3 The concepts of a dual strong sequence and dual strong union

Let $\langle E, F\rangle$ be a dual pair. Starting with $F = G_1$ and $\beta_1 = \beta(E, G_1)$ and keeping on with $G_2 = (E, \beta_1)'$ and $\beta_2 = \beta(E, G_2)$ we perceive a structure described in the following definition.

Definition 3.1. Let $\langle E, F\rangle$ be a dual pair. We say that $(G_n)_{n\in N}$ is the *dual strong sequence* of $\langle E, F\rangle$ if $G_1 = F, \beta_n = \beta(E, G_n), G_{n+1} = (E, \beta_n)'$, for every positive integer $n \in \mathbb{N}$. We say that $G = \cup\{G_n : n \in \mathbb{N}\}$ is *the dual strong union* of $\langle E, F\rangle$.

The space E and its dual strong union G constitute a dual pair $\langle E, G\rangle$, enabling to generate a new dual strong sequence, which leads us to the next definition.

Definition 3.2. Let $\langle E, F\rangle$ be a dual pair. Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Consider the following structure on E^* for $k \in \mathbb{N}_0, n \in \mathbb{N}$:

(1) $F = G_{01}$,

(2) $\beta_{kn} = \beta(E, G_{kn})$,

(3) $G_{k,n+1}$ is the dual of (E, β_{kn}),

(4) $G_{k+1,1} = \cup\{G_{kn} : n \in \mathbb{N}\}$.

We say that $(G_{kn})_{n\in\mathbb{N}}$ is *the k-th dual strong sequence* and $G_{n,1}$ *the n-th dual strong union* of $\langle E, F\rangle$. We say that $(G_{0n})_{n\in\mathbb{N}}$, resp. G_{11}, is the *initial* dual strong sequence, resp. union, of $\langle E, F\rangle$ and denote $(G_{0n})_{n\in\mathbb{N}} = (G_n)_{n\in\mathbb{N}}, G_{11} = G$.

Remark 3.1. In the context of Definition 3.2, the following holds for $k \in \mathbb{N}_0, n \in \mathbb{N}$,

(a) the set of subspaces (G_{kn}) of E^* is (well) ordered by inclusion :

$$G_{kn} \subseteq G_{mp} \quad \text{if and only if} \quad ((k < m) \vee ((k = m) \wedge (n \leq p)));$$

(b) $\beta_{kn} \leq \beta_{mp}$ for any k, m, n, p, satisfying : $((k < m) \vee ((k = m) \wedge (n \leq p)))$;

(c) the n-th dual strong sequence of $\langle E, F\rangle$ is the initial dual strong sequence of $\langle E, G_{n-1,1}\rangle$ for any $n \in \mathbb{N}$.

Definition 3.3. The ordered set (G_{kn}) of Definition 3.2 is named *the generalized dual strong sequence of $\langle E, F\rangle$*. The set $\cup\{G_{kn} : k \in \mathbb{N}_0, n \in \mathbb{N}\} = \cup\{G_{k,1} : k \in \mathbb{N}\}$ is *the generalized dual strong union of $\langle E, F\rangle$*.

Remark 3.2. Let (G_{kn}) be the generalized dual strong sequence of $\langle E, F\rangle$.

(a) $G_{k,n+1}$ is the bidual of $(G_{kn}, \mu(G_{kn}, E))$, for any $k \in \mathbb{N}_0, n \in \mathbb{N}$.

(b) For any fixed pair $(k, n), k \in \mathbb{N}_0, n \in \mathbb{N}$, the tail $\{G_{mp} : m \in \mathbb{N}_0, p \in \mathbb{N}\}$, such that $((k < m) \vee ((k = m) \wedge (n \leq p)))\}$ is the generalized dual strong sequence of $\langle E, G_{kn}\rangle$.

Definition 3.4. A subspace L is *boundedly completed (quasi-distinguished) in (E, t)*, if any element (bounded set) of (E, t) of contained in the closure of a bounded subset of (L, t). The space (E, t) is *boundedly completed (quasi-distinguished)*, if it is boundedly completed (quasi distinguished) in the completion of (E, t).

Boundedly completed and quasi-distinguished spaces were defined and investigated in ([29], [31], [33]). Certainly a quasi-distinguished subspace of E is also boundedly completed in E. There exists an example of a metrizable, hence boundedly completed, but not quasi-distinguished locally convex space, [2]. Any subspace of a locally convex Fréchet-Urysohn space (i.e. a space in which a subset is closed if and only if it is sequentially closed) is boundedly completed. A discussion and examples of nonmetrizable Fréchet-Urysohn spaces can be found in [7].

Remark 3.3. Given the k-th dual strong sequence of $\langle E, F\rangle, G_{kn}$ is boundedly completed in $(G_{k,n+1}, \sigma(G_{k,n+1}, E)), k \in \mathbb{N}_0, n \in \mathbb{N}$.

The next remark states that the barrelledness of E is the "bottom line" for the dual strong sequence of $\langle E, F\rangle$. Recall that a locally convex space E is *distinguished*, if $\beta(E', E) = \beta(E', E'')$, or equivalently, if $(E', \beta(E', E))$ is barrelled, [18].

Remark 3.4. Using the terms of Definition 3.2, the following statements are equivalent :

(i) there exist $k \in \mathbb{N}_0, n \in \mathbb{N}$ such that G_{kn} is quasi-distinguished in $(G_{k,n+1}, \sigma(G_{k,n+1}, E))$;

(ii) there exist $k \in \mathbb{N}_0, n \in \mathbb{N}$ such that $(G_{kn}, \mu(G_{kn}, E))$ is distinguished;

(iii) there exist $k \in \mathbb{N}_0, n \in \mathbb{N}$ such that $G_{mp} = G_{hr}$ for positive integers m, h, p, r, satisfying : $m, h > k$ and $p, r > n$;

(iv) there exist $k \in \mathbb{N}_0, n \in \mathbb{N}$ such that $\beta_{mp} = \beta_{hr}$ for positive integers m, h, p, r, satisfying $m, h > k$ and $p, r > n$;

(v) there exist $k \in \mathbb{N}_0, n \in \mathbb{N}$ such that (E, β_{kn}) is barrelled.

We conclude this section with the following "perpetuum mobile" observation.

Remark 3.5. The formal process of generating dual strong sequences and dual strong unions can be initialized and continued by substituting $F = \cup\{G_{kn} : k \in \mathbb{N}_0, n \in \mathbb{N}\} = \cup\{G_{k,1} : k \in \mathbb{N}\}$ and using the steps of Definition 3.2.

4 Bounded sets of the dual strong sequence and dual strong union

For different families $\mathbf{A}, \mathbf{B}$ of bounded disks in a locally convex space E we define the usual (partial) order relation : $\mathbf{A} \leq \mathbf{B}$ if and only if for any $A \in \mathbf{A}$ there exists $B \in \mathbf{B}$ such that $A \subseteq B$. If $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A}$ we write $\mathbf{A} \sim \mathbf{B}$.

For $F_1 \subseteq F_2 \subseteq E^*$, a $\sigma(E, F_1)$-closed bounded disk B of E is $\sigma(E, F_2)$-closed but not necessarily bounded. In a similar fashion, a $\sigma(E, F_2)$-closed bounded disk B is bounded but not necessarily closed in $(E, \sigma(E, F_1))$. By using $\mathbf{A} \subseteq \mathbf{B}$ ($\mathbf{A} = \mathbf{B}$) instead of $\mathbf{A} \leq \mathbf{B}$ ($\mathbf{A} \sim \mathbf{B}$) we indicate that there is no closure related confusion and exactly the same closed bounded disks participate in both $\mathbf{A}$ and $\mathbf{B}$.

The next proposition establishes the connections between the families of type 1, 2, 5, 6 for the generalized dual strong sequence of $\langle E, F\rangle$. The families of type 3, 4 will be treated in the forthcoming article. For a generalized dual strong sequence $(G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$, we denote by $\mathbf{A}_{kn}$ a family of bounded disks in $(E, \sigma(E, G_{kn}))$.

Proposition 4.1. *Let $(G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$ be the generalized dual strong sequence of $\langle E, F\rangle$. Then for each $n \in \mathbb{N}$ the following properties hold :*

(a) $\mathbf{B}_{kn} \supseteq \mathbf{B^*}_{kn} = \mathbf{B}_{k,n+1}$,

(b) $\mathbf{B^*}_{kn} \geq \mathbf{B^*}_{k,n+1}$,

(c) $\mathbf{WC}_{kn} \supseteq \mathbf{WC}_{k,n+1}$,

(d) $\mathbf{FIN}_{kn} = \mathbf{FIN}_{k,n+1}$.

Proof. (a) Obviously $\mathbf{B}_{kn} \supseteq \mathbf{B^*}_{kn}$. If $B \in \mathbf{B^*}_{kn}$, then B is closed and bounded in (E, β_{kn}). The topology β_{kn} is compatible with the duality $\langle E, G_{k,n+1}\rangle$, hence $B \in \mathbf{B}_{k,n+1}$. On the other side, if $B \in \mathbf{B}_{k,n+1}$ then B is bounded for (E, β_{kn}) whose 0-nbhd base consists of $\sigma(E, G_{kn})$-closed subsets, namely the polars in E of all bounded disks of G_{kn}. Therefore the closure of B in $(E, \sigma(E, G_{kn}))$ belongs to $\mathbf{B^*}_{kn}$ and to $\mathbf{B}_{k,n+1}$ as well, hence $\mathbf{B^*}_{kn} = \mathbf{B}_{k,n+1}$.

(b) If $B \in \mathbf{B^*}_{k,n+1}$, then B° in $G_{k,n+1}$ is a $\sigma(G_{k,n+1}, E)$-bornivorous barrel, therefore $B^\circ \cap G_{kn}$ is a bornivorous barrel in $(G_{kn}, \sigma(G_{kn}, E))$, hence $(B^\circ \cap G_{kn})^\circ$ being the $\sigma(E, G_{kn}))$-closure of B belong to $\mathbf{B^*}_{kn}$, therefore $\mathbf{B^*}_{kn} \geq \mathbf{B^*}_{k,n+1}$.

(c) A compact disk is compact in any coarser topology, hence the conclusion.

(d) is obvious. □

Loosely speaking, when the chain (G_{kn}) increases in the sense of Remarks 3.1 (a) and 3.5, the bounded families $\mathbf{B}_{kn}$ of $(E, \sigma(E, G_{kn}))$ shrink, however restrained by the **FIN** family of E. The members of $\mathbf{WC}_{kn}$ family of E change their status, becomming noncompact members of the $\mathbf{BAN}_{k,n+1}$ family. On the other side the bounded families of $(G_{kn}, \sigma(G_{kn}, E))$ flourish, their expansion being restricted by Remark 3.4 and E^*.

The following proposition describes the structure of bounded sets in the dual strong union.

Proposition 4.2. *Let $\langle E, F\rangle$ be a dual pair, $(G_n)_{n\in\mathbb{N}}$ the initial dual strong sequence, and G the initial dual strong union. Any closed bounded disk in $(G, \sigma(G, E))$ is either compact or a countable union of an increasing sequence of compact disks in $(G, \sigma(G, E))$.*

Proof. If a bounded disk A is contained in some G_n then the closure of A in $(G_{n+1}, \sigma(G_{n+1}, E))$ is weakly compact in G_{n+1} and therefore in G. Otherwise for a $\sigma(G, E)$-closed bounded disk A we denote $A_n = A \cap G_n$. Then $A = \cup\{A_n : n \in \mathbb{N}\}$. Since A is closed in $(G, \sigma(G, E))$, the disk A_n is closed in $(G_n, \sigma(G_n, E))$ for any $n \in \mathbb{N}$. If $\bar{A}_n$ is the closure of A_n in $(G_{n+1}, \sigma(G_{n+1}, E))$, we have $A_n \subseteq \bar{A}_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$ and therefore $A = \cup\{\bar{A}_n : n \in \mathbb{N}\}$. Since A_n is $\sigma(G_n, E)$-bounded, $\bar{A}_n$ is $\sigma(G_{n+1}, E)$-compact, hence $\sigma(E^*, E)$-complete and therefore compact in $(G, \sigma(G, E))$. We proved that A is a countable union of an increasing sequence $(\bar{A}_n)$ of compact disks in $(G, \sigma(G, E))$. □

Proposition 4.2 and Remark 3.1 (c) yeld the following result.

Proposition 4.3. *Let G_{n1} be the n-th dual strong union of a dual pair $\langle E, F\rangle$. The following statements hold for each $n \in \mathbb{N}$.*

(a) Any closed bounded disk in $(G_{n1}, \sigma(G_{n1}, E))$ is either compact or a countable union of an increasing sequence of compact disks.

(b) Any barrel in $(E, \mu(E, G_{n1}))$ is either a 0-nghb or a countable intersection of a decreasing sequence of 0-nghb.

Proof. (a) Follows from Proposition 4.2 and Remark 3.1 (c).

(b) If $A = \cup\{\bar{A}_n : n \in \mathbb{N}\}$ is a closed bounded disk in G_{n1}, then $A^\circ = \cap\{\bar{A}_n^\circ : n \in \mathbb{N}\}$ is a barrel and $\bar{A}_n^\circ$ is a 0-nghb in $(E, \mu(E, G_{n1}))$. □

5 Barrelledness and the dual strong union

Suppose $\{U_n : n \in \mathbb{N}\}$ is a sequence of barrels in E such that every $x \in E$ belongs to all but a finite number of U_n. Then $U = \cap\{U_n : n \in \mathbb{N}\}$ is a barrel in E. According to Mazon, E is *C-barrelled* if for any sequence $\{U_n : n \in \mathbb{N}\}$ of absolutely convex closed 0-nghb such that for every $x \in E$ there exists $n_x \in \mathbb{N}$ such that $x \in U_n$ for each $n \geq n_x$, the barrel $U = \cap\{U_n : n \in \mathbb{N}\}$ is a 0-nghb, ([23]), Definition 8.2.6). According to Webb (E, τ) is *c_0-barrelled* if any weakly convergent to zero sequence in $(E, \tau)'$ is τ-equicontinuous, (Webb used the term *sequentially barrelled*, see [38]). According to Ruess, (E, τ) *has property* (L) if for any absorbing disk A, τ is the finest locally convex topology that coincides on A with τ. Ruess mentioned that property (L) "is a certain weakening of barrelledness", ([27], p. 180). Saxon and Sanchez Ruiz affiliated the dual local completeness with a weak barrelledness condition by proving that a dual locally complete Mackey space is C-barrelled, ([28], Theorem 3.2). Qiu consolidated the collection of barrelled properties by observing the equivalency of C-barrelledness, dual local completeness, property (L) and c_0-barrelledness, extending the equivalence results to the quasi-barrelled case as well, ([24], Theorems 4 and 5).

The next proposition glues the **NBAR, BAN** and **WC** families of the dual strong union. We need the following result of Valdivia : ([35], Theorem 6). *Let $\{U_n : n \in \mathbb{N}\}$ be an increasing sequence of closed disks in a barrelled space E such that $E = \cup\{U_n : n \in \mathbb{N}\}$. Then any bounded set is absorbed by some U_n.*

Proposition 5.1. *Let $\langle E, F\rangle$ be a dual pair and G_{n1} the n-th dual strong union for some $n \in \mathbb{N}$. If B is a closed bounded disk in $(G_{n1}, \sigma(G_{n1}, E))$ such that E_B is barrelled then B is weakly compact.*

Proof. Let B be a closed disk in $(G_{n1}, \sigma(G_{n1}, E))$. By Proposition 4.3, B is either compact or $B = \cup\{\bar{A}_n : n \in \mathbb{N}\}$, each $\bar{A}_n$ being $\sigma(G_{n1}, E)$-compact. If $B = \cup\{\bar{A}_n : n \in \mathbb{N}\}$ then $E_B = \cup\{n\bar{A}_n : n \in \mathbb{N}\}$ is a normed barrelled space satisfying the conditions of ([35], Theorem 6). Hence B is absorbed by some $\bar{A}_n$ therefore B is $\sigma(G_{n1}, E)$-compact. □

The proposition we just proved claims that **NBAR** = **BAN** = **WC** in any n-th dual strong union $(G_{n1}, \sigma(G_{n1}, E))$, allowing to join the Ruess remark of ([27], p. 180) by suggesting the dual locally (quasi) barrelledness is a certain weakening of (quasi) barrelledness.

Proposition 5.2. *Let $\langle E, F\rangle$ be a dual pair and G_{n1} the n-th dual strong union for some $n \in \mathbb{N}$. The following statements are equivalent :*

(i) $(E, \mu(E, G_{n1}))$ *is dual locally barrelled,*

(ii) $(E, \mu(E, G_{n1}))$ *is dual locally complete (or dual ℓ^q-complete, see [24]),*

(iii) $(E, \mu(E, G_{n1}))$ *is c_0-barrelled,*

(iv) $(E, \mu(E, G_{k,1}))$ *has property (L),*

(v) $(E, \mu(E, G_{n1}))$ *is C-barrelled,*

(vi) $(E, \mu(E, G_{n1}))$ *is barrelled.*

Proof. (i) $\Rightarrow$ (vi) : follows from Proposition 5.1. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) : see the discussion on page 1423 and Theorem 4 of [24]. (vi) $\Rightarrow$ (ii) $\Rightarrow$ (i) : obvious. □

We say that $(A_n)_{n\in\mathbb{N}}$ is an *absorbent sequence* in a locally convex space E if it is an increasing sequence of disks whose union $A = \cup\{A_n : n \in \mathbb{N}\}$ is absorbing. It is *bounded-absorbent* if any bounded subset of E is absorbed by some A_n.

In [9, Theorem 1] De Wilde and Houet proved that given a dual pair $\langle E, F\rangle$, any absorbent sequence of closed disks in E is bounded-absorbent for $(E, \beta^*(E, F))$. Combining ([9], Theorem 1) with a mild restriction of having the Banach-Mackey property in E or equivalently in F, we conclude that *in a Banach-Mackey space any absorbent sequence of closed disks is bounded-absorbent.*

If E admits an absorbent sequence $(A_n)_{n\in\mathbb{N}}$ of closed disks, then $E = \cup\{nA_n : n \in \mathbb{N}\}$ allowing the UBL related inquiry, ([34], see also [23]). Indeed the result of De Wilde-Houet activated a vigorous research, inspiring new concepts, such as the property (B) of Ruess, (see the italicized conclusion), or the properties (L) and (Lb) (see [27], p. 180). Given a dual pair $\langle E, F\rangle$, Theorem 1 of [9] amalgamates the following concepts : E is Banach-Mackey $\Leftrightarrow$ E has property (B) $\Leftrightarrow$ F is Banach-Mackey $\Leftrightarrow$ F has property (B). The paper [9] of De Wilde and Houet appeared almost simultaneously with the famous paper [35] of Valdivia, where similar results were established for barrelled spaces. Our next proposition is based on De Wilde-Houet result. In order to be self-content we repeat the original proof of [9]. The notations of [9] follow the tradition of [17]. We change some of them in the spirit of [22] and [23].

Theorem 5.1 (De Wilde-Houet). *Given a dual pair $\langle E, F\rangle$, any absorbent sequence of closed subsets of $(E, \sigma(E, F))$ is bounded-absorbent in $(E, \beta^*(E, F))$.*

Proof. (see [9], p. 257). Let $\{A_n : n \in \mathbb{N}\}$ be an absorbent sequence of closed sets in $(E, \sigma(E, F))$ and B is a subset of E. Suppose $B \not\subseteq nA_n$ for all $n \in \mathbb{N}$. Then there exist $x_n \in B$ and $f_n \in A_n^\circ$ such that $|f_n(x_n)| > n$. Since $\{A_n : n \in \mathbb{N}\}$ is increasing and $\cup\{A_n : n \in \mathbb{N}\}$ is absorbing, $\{f_n : n \in \mathbb{N}\}$ is $\sigma(F, E)$-bounded. If B is strongly bounded, then B° is a bornivorous barrel in F, hence B° absorbs the set $\{f_n : n \in \mathbb{N}\}$. But $\{x_n : n \in \mathbb{N}\} \subseteq B$ and $\{f_n : n \in \mathbb{N}\}$ is unbounded on $\{x_n : n \in \mathbb{N}\}$, hence the contradition, therefore B is absorbed by some A_n. □

Remind that $\mathbf{B}_{kn}$, resp. $\mathbf{B}^*{}_{kn}$ is the family of all closed bounded, resp. strongly bounded, disks in $(E, \sigma(E, G_{kn}))$.

Proposition 5.3. *Let $(G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$ be the generalized dual strong sequence of $\langle E, F\rangle$. Any absorbent sequence of closed disks in $(E, \sigma(E, G_{kn}))$ for some $k \in \mathbb{N}_0, n \in \mathbb{N}$ is bounded-absorbent in $(E, \sigma(E, G_{mp}))$ for any m, p, satisfying $G_{k,n+1} \subseteq G_{mp}$ (i.e. for the tail of $G_{k,n+1}$).*

Proof. Let $\{A_r : r \in \mathbb{N}\}$ be an absorbent sequence of closed disks in $(E, \sigma(E, G_{kn}))$ for some $k \in \mathbb{N}_0, n \in \mathbb{N}$. Then any $B \in \mathbf{B}^*{}_{kn}$, is absorbed by some A_r. Hence by Proposition 4.1 (a) any $B \in \mathbf{B}_{k,n+1}$ is absorbed by some A_r. The conclusion follows by noticing that $\mathbf{B}_{k,n+1} \geq \mathbf{B}_{mp}$ and the disks $\{A_r : r \in \mathbb{N}\}$ are closed in $(E, \sigma(E, G_{mp}))$, for any tail G_{mp} of $G_{k,n+1}$. □

To stress the significance of the generalized dual strong sequence we reformulate Proposition 5.3 and obtain :

Proposition 5.4. *Let $(G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$ be the generalized dual strong sequence of $\langle E, F\rangle$. Any absorbent sequence of closed disks in $(E, \sigma(E, F))$ is bounded-absorbent in $(E, \sigma(E, G_{kn})), n > 1$.*

6 The scope of the associated barrelled topology

Given a locally convex nonbarrelled space (E, τ), we say that τ_{bar} is *the associated barrelled topology* for τ if τ_{bar} is the weakest barrelled topology, finer than τ. If (E, τ) is barrelled, then $\tau_{bar} = \tau$. The next proposition extends Theorem 1 of [31].

Proposition 6.1. *Let $\langle E, F\rangle$ be a dual pair, $(G_n)_{n \in \mathbb{N}}$ the initial dual strong sequence, G the initial dual strong union and τ_{bar} the associated barrelled topology for $\mu(E, F)$. Then τ_{bar} is the associated barrelled topology for β_n and for $\beta(E, G), n \in \mathbb{N}$.*

Proof. If $(E, \mu(E, F))$ is barrelled, then $\tau_{bar} = \mu(E, F)$ and $F = G_n$ for each $n \in \mathbb{N}$, therefore $\tau_{bar} = \mu(E, F) = \beta_n$. For a nonbarrelled $(E, \mu(E, F))$ we notice that $\tau_{bar} \geq \beta_1 = \beta(E, F)$. By Remark 3.3, $F = G_1$ is boundedly completed in $(G_2, \sigma(G_2, E))$, therefore we have $\beta_1 \geq \sigma(E, G_2)$ which implies $\tau_{bar} \geq \sigma(E, G_2)$. But then $\tau_{bar} \geq \mu(E, G_2)$, hence $\tau_{bar} \geq \beta(E, G_2) = \beta_2$. Since G_n is boundedly completed in $(G_{n+1}, \sigma(G_{n+1}, E))$, we conclude by induction that $\tau_{bar} \geq \beta_n$ for any positive integer n.

Let $\tau_{G\,bar}$ be the associated barrelled topology for $\mu(E, G)$. Consider the duals $(E, \tau_{bar})' = F_{bar}, (E, \tau_{G\,bar})' = G_{bar}$. If for some $k \in \mathbb{N}$, (E, β_k) is barrelled, Remark 3.4 provides $G_n = G = F_{bar}$ for each $n > k$, therefore $\tau_{bar} = \beta_k = \mu(E, G) = \tau_{G\,bar}$. If (E, β_n) is nonbarrelled for each $n \in \mathbb{N}$, then $G_n \subseteq F_{bar}$ and therefore $G \subseteq F_{bar}$. It follows, that $\tau_{bar} \geq \mu(E, G)$, and therefore $\tau_{bar} \geq \tau_{G\,bar}$. On the other hand, $F \subseteq G$, therefore $F_{bar} \subseteq G_{bar}$, hence $\tau_{bar} \leq \tau_{G\,bar}$, and we conclude that $\tau_{bar} = \tau_{G\,bar}$. □

The next proposition illuminates the horizon of the associated barrelled topology.

Proposition 6.2. *Let $(G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$, be the generalized dual strong sequence for a dual pair $\langle E, F\rangle$. Let τ_{bar} be the associated barrelled topology for $\mu(E,F)$. Then τ_{bar} is the associated barrelled topology for any $\beta(E, G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$.*

Proof. By Proposition 6.1, τ_{bar} is the associated barrelled topology for $\beta(E, G_{11}), n \in \mathbb{N}$. Suppose that for some $k \in \mathbb{N}$ the topology τ_{bar} is the associated barrelled topology for $\beta(E, G_{kn})$ and $\beta(E, G_{k+1,1}), n \in \mathbb{N}$. Then keeping in mind Remark 3.1 (c) and using the arguments of Proposition 6.1 we conclude that τ_{bar} is the associated barrelled topology for $\beta(E, G_{k+1,n})$ and $\beta(E, G_{k+2,1}), n \in \mathbb{N}$. □

Combining Remark 3.4 with Proposition 6.2 we obtain :

Proposition 6.3. *In the setting of Proposition 6.2, the following is equivalent.*

(i) τ_{bar} *is the associated barrelled topology for* $\mu(E,F)$;

(ii) τ_{bar} *is the associated barrelled topology for any* $\mu(E, G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$;

(iii) τ_{bar} *is the associated barrelled topology for any* $\beta(E, G_{kn}), k \in \mathbb{N}_0, n \in \mathbb{N}$.

7 A remark on Perez Carreras - Bonet - Qiu completeness scheme

In [24] Qui extends the completeness scheme of Perez Carreras and Bonet [23], describing the relationship between different notions of completeness, each connected to some kind of compactness and/or barrelledness. We suggest to amplify the scheme of Qiu with the notions of B-completeness, B_r-completeness and local barrelledness. A locally convex space E is B-complete (B_r-complete) if every (dense) subspace $L \subseteq E'$ with a closed intersection $L \cap U^\circ$ for any closed equicontinuous disk U° of E', is itself closed in E' (coincides with E'). The B- and B_r-completeness are related to the minimal barrelled topologies via the Closed Graph - Open Mapping Theorems of Ptak. We also mention the correspondence between the barrelled topologies on E and its closed subspaces, (see [11], [13], [15]). As to the local barrelledness notion, we believe that Propositions 5.1 and 5.2 of this article justify the inclusion.

Qiu [24] begins the scheme with completeness and concludes with local completeness. Seeing the revised scheme in the comprehensive context of Baire categorization, we suggest starting from B-completeness and closing with the local barrelledness.

$$B\text{-complete} \Rightarrow B_r\text{-complete} \Rightarrow \text{complete} \Rightarrow \text{continuing as in [24]} \Leftrightarrow \ell c \Rightarrow \ell b.$$

The newly introduced parts of the scheme are irreversible. Examples of complete but not B_r-complete spaces are trivial, along with examples of locally barrelled but not locally complete spaces. An example of a B_r-complete space which is not B-complete was obtained by Valdivia in [37].

8 Questions

We conclude this article with some questions and suggestions for further research. It seems natural to investigate the conditions for terminating the generalized dual strong sequence in specific locally convex spaces, thus attaining the associated barrelled topology. It also seems promising to combine the fine-grained research indepth of compactness and biduality in tangible locally convex spaces with the general results of this article (see [3], [4], [8], [10] for insights and references).

Given a dual pair $\langle E, F\rangle$, is seems natural to search the inheritance of different properties on E by bidual enlargements of F, for example, Banach-Mackey or separability. It looks interesting because the bounded sets can be drastically transformed by biduals (see [14] as an illustration).

In view of Proposition 4.3 another reasonable question arises : does a closed bounded disk in the dual strong union possess a quasi-weak drop property ? We refer to [19] and [25] for the necessary setting, definitions and references.

The author is very grateful to Professor J. Schmets for valuable remarks and suggestions after reading the manuscript.

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