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Hypercyclicity criteria and the Mittag-Leffler theorem

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Dedicated to the memory of Pascal Laubin.

Abstract

We show that different hypercyclicity criteria are equivalent by using the abstract version of Mittag-Leffler theorem. We also reduce to the context of invertible operators an open problem of Herrero which asks about the hypercyclicity of the direct sum of a hypercyclic operator with itself. [1]

1 Introduction

One of the "wildest" behaviours that a linear operator $T : E \to E$ can exhibit is the existence of vectors $x \in E$ whose orbit $\mathrm{Orb}(T, x) := \{x, Tx, T^2x, \dots\}$ is dense in E. In such a case T is called *hypercyclic* and x is a hypercyclic vector for T. This is only allowed for infinite dimensional spaces E (see, e.g., [7] and [4]). The first example was given by Birkhoff [6] who showed that the translation operator $T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $f(z) \mapsto f(z+a)$, $(a \neq 0)$ is hypercyclic on the Fréchet space $\mathcal{H}(\mathbb{C})$ of entire functions endowed with the compact-open topology. Later, MacLane [13] proved the hypercyclicity of the differentiation operator $D : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $f \mapsto f'$. The first example of a hypercyclic operator on a Banach space was given by Rolewicz [15] showing that the weighted backward shift $\lambda B : l_p \to l_p$, $(x_1, x_2, \dots) \mapsto (\lambda x_2, \lambda x_3, \dots)$ is hypercyclic if $|\lambda| > 1$. All these proofs were direct, but probably the argument of Rolewicz indicated the possibility to give some kind of general criterion under which an operator is hypercyclic. This criterion was finally found by Kitai [12] in her unpublished Phd Thesis. Later, Gethner and Shapiro [8] rediscovered the criterion. Since then many hypercyclicity criteria have been given and our intention here is to unify these criteria by using a Mittag-Leffler argument.

Our framework will be continuous linear operators $T : E \to E$ $(T \in L(E))$ on $\mathcal{F}$-spaces (i.e., complete and metrizable topological vector spaces) E. The following is, essentially, the original hypercyclicity criterion (see [8, Thm. 2.2 and Remarks 2.3]):

Theorem 1.1 (Kitai/Gethner-Shapiro) *Let E be a separable $\mathcal{F}$-space and $T \in L(E)$. If there are dense subsets $X, Y \subset E$, a map $S : Y \to Y$, and an increasing sequence (n_k) of natural numbers such that*

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(i) $T^{n_k}x \xrightarrow{k} 0$, *for any* $x \in X$,

(ii) $S^{n_k}y \xrightarrow{k} 0$, *for any* $y \in Y$, *and*

(iii) $T \circ S = I_Y$, *then*

T *is hypercyclic. In fact* T *admits a dense* G_δ*-set of hypercyclic vectors.*

The hypothesis easily imply a property known as *topological transitivity*, namely, for every $U, V \subset E$ non-empty open sets, there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. The rest uses a Baire argument.

This criterion might seem a bit too technical, but it is easy to compute in concrete operators. This is the case for Rolewicz's example by setting

$$X = Y := \{x = (x_k) \ / \ \exists j \ : \ x_k = 0 \ \forall k \geq j\}$$

and $S : Y \to Y$, $(x_1, x_2, \dots) \mapsto (0, \frac{1}{\lambda}x_1, \frac{1}{\lambda}x_2, \dots)$; or MacLane's differentiation operator if we set $X = Y := \{polynomials\}$, $S(z^n) := \frac{z^{n+1}}{n+1}$, extended by linearity to Y. In the case of Birkhoff's example, Godefroy and Shapiro [9] proved the hypercyclicity of T_a by considering the dense subspaces of $\mathcal{H}(\mathbb{C})$:

$$X := \text{span}\{e^{\lambda z} \ : \ |e^{\lambda a}| < 1\}, \quad Y := \text{span}\{e^{\lambda z} \ : \ |e^{\lambda a}| > 1\},$$

and $S := T_{-a}$.

Juan Bès observed [3] (see also [5]) that the hypothesis of the hypercyclicity criterion can be substantially relaxed. More precisely, there is no need to impose the existence of a right inverse S of T on a dense subspace of E. This is what today is known as the *Hypercyclicity Criterion*:

Theorem 1.2 (Bès) *Let* E *be a separable* $\mathcal{F}$*-space and* $T \in L(E)$. *If there are dense subsets* $X, Y \subset E$, *an increasing sequence* (n_k) *of natural numbers, and a sequence* $\{S_{n_k} : Y \to E \ , \ k \in \mathbb{N}\}$ *of maps such that*

(i) $T^{n_k}x \xrightarrow{k} 0$, *for any* $x \in X$,

(ii) $S_{n_k}y \xrightarrow{k} 0$, *for any* $y \in Y$, *and*

(iii) $(T^{n_k} \circ S_{n_k})y \xrightarrow{k} y$, *for any* $y \in Y$, *then*

T *is hypercyclic.*

Bès's criterion is related to a problem of Herrero [11]: *Does every hypercyclic operator* T *satisfy that* $T \oplus T$ *is also hypercyclic?* To be precise, T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic [5, Thm. 2.3]. Therefore, Herrero's problem is equivalent to the open problem of whether every hypercyclic operator satisfies the Hypercyclicity Criterion. We reduce this problem to the context of invertible operators.

Other criteria have been given by Grosse-Erdmann [10]. In [2] we showed that they are actually equivalent to Bès's criterion. In this paper we close the circle and prove their equivalence to the original criterion.

We will use the following abstract version of Mittag-Leffler theorem (see, e.g., [1, Thm. 2.4]).

Theorem 1.3 (Mittag-Leffler) *Let* (X_n) *be a sequence of complete metric spaces and let* $f_n : X_{n+1} \to X_n$, $n \in \mathbb{N}$, *be continuous maps with dense range. Then, for every non empty open subset* $U \subset X_1$, *there exists a sequence* $\{x_n \in X_n \ , \ n \in \mathbb{N}\}$ *such that* $x_1 \in U$ *and* $f_n(x_{n+1}) = x_n$, $n \in \mathbb{N}$.

2 Main results

¿From now on E will be an $\mathcal{F}$-space and $T : E \to E$ a continuous linear operator with dense range. This condition is necessary for hypercyclic operators. We consider the subspace of vectors which admit a backward orbit (also known as the infinite core of T),

$$F := \{x \in E \ / \ \exists (x_i) \subset E \ : \ x_1 = x, \ Tx_{i+1} = x_i, \ \forall i \in \mathbb{N}\}.$$

It is well known that F is a dense subspace of E. Indeed, it is a consequence of Mittag-Leffler theorem if we set $X_n := E$, $f_n := T$, for all $n \in \mathbb{N}$.

The *space of backward orbits* of T is then defined as

$$G := \{(x_i) \in \prod_{i \in \mathbb{N}} E \ / \ Tx_{i+1} = x_i, \ \forall i \in \mathbb{N}\}.$$

We endow G with the inherited product topology and it is an $\mathcal{F}$-space. In fact G is the projective limit of the projective spectrum of $\mathcal{F}$-spaces

$$(X_i := E, \ f_{i,j} := T^{j-i} : X_j \to X_i)_{i \leq j \in \mathbb{N}}.$$

Moreover, T induces a natural operator $\tilde{T} : G \to G$, $(x_i) \mapsto (Tx_i)$. Observe that $\tilde{T}(x_i) = (Tx_1, x_1, x_2, \dots)$ and that $\tilde{T}$ is an invertible operator whose inverse is the backward shift $B : G \to G$, $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$.

The following proposition links the dynamics of T and $\tilde{T}$ and it is the key point to establish the main results.

Proposition 2.1 *$T : E \to E$ is hypercyclic if and only if the invertible operator $\tilde{T} : G \to G$ is hypercyclic.*

Proof. Let (U_m) be a basis of open neighbourhoods of 0 (0-basis in short) in E. A simple computation shows that a 0-basis in G is given by the sequence

$$\tilde{U}_{n,m} := \{(x_i) \in G \ / \ x_n \in U_m\}, \quad n, m \in \mathbb{N}.$$

Let suppose that T is hypercyclic. For the first implication it suffices to show that $\tilde{T}$ is topologically transitive. To do this we have to prove that, given $(x_i), (y_i) \in G$, $n, m \in \mathbb{N}$, there is $k \in \mathbb{N}$ satisfying

$$\tilde{T}^k \left((x_i) + \tilde{U}_{n,m}\right) \cap \left((y_i) + \tilde{U}_{n,m}\right) \neq \emptyset.$$

We know that there is $k \in \mathbb{N}$ such that

$$T^k(x_n + U_m) \cap (y_n + U_m) \neq \emptyset.$$

The density of F in E gives

$$(x_n + U_m \cap F) \cap (T^k)^{-1}(y_n + U_m) = (x_n + U_m) \cap (T^k)^{-1}(y_n + U_m) \cap F \neq \emptyset.$$

Then there is $z \in U_m \cap F$ such that

$$(*) \quad T^k(x_n + z) \in y_n + U_m.$$

Pick z_i, $i > n$, so that $T^{i-n} z_i = z$, and set $z_j := T^{n-j} z$, $j = 1, \ldots, n$. Therefore $(z_i) \in \tilde{U}_{n,m}$ and $(*)$ yields

$$\tilde{T}^k \left((x_i) + (z_i) \right) \in (y_i) + \tilde{U}_{n,m}.$$

Conversely, if $\tilde{T}$ is hypercyclic, we consider the first coordinate projection $P_1 : G \to E$ and the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{T}} & G \\ {\scriptstyle P_1}\big\downarrow & & {\scriptstyle P_1}\big\downarrow \\ E & \xrightarrow{T} & E \end{array}$$

where P_1 has dense range since $F = P_1(G)$. The conclusion follows from [14, Lemma 2.1]. ■

Remarks:

(1) The space F becomes an $\mathcal{F}$-space when defining on it the final topology τ associated to $P_1 : G \to F$. A 0-basis in (F, τ) is given by the sequence

$$V_{n,m} := T^n (U_m \cap F), \qquad n, m \in \mathbb{N}.$$

We then have that $T|_F : (F, \tau) \to (F, \tau)$ is continuous and that the inclusion $(F, \tau) \to E$ is continuous and has dense range. Moreover the proof of last proposition shows that the surjective operator $T|_F : (F, \tau) \to (F, \tau)$ is hypercyclic whenever T is hypercyclic. This should be compared with the Hypercyclicity Comparison Principle (see [16, p.111]) which says that, if F is a dense and continuously embedded subspace of E, and $T : E \to E$ is an operator such that F is T-invariant and the restriction $T|_F$ is hypercyclic, then T is hypercyclic.

(2) Instead of the space of backward orbits we could have defined the *space of full orbits* of T

$$H := \{(x_i) \in \prod_{i \in \mathbb{Z}} E \ / \ T x_{i+1} = x_i, \ \forall i \in \mathbb{Z}\}.$$

In such a case H is the projective limit of the projective spectrum of $\mathcal{F}$-spaces

$$(X_i := E, \ f_{i,j} := T^{j-i} : X_j \to X_i)_{i \le j \in \mathbb{Z}},$$

and the operator $\tilde{T} : H \to H$ induced by T is nothing but the forward shift.

We analogously have that T is hypercyclic if and only if $\tilde{T} : H \to H$ is hypercyclic.

Proposition 2.1 allows us to restrict Herrero's problem to the context of invertible operators

Theorem 2.2 *There exists a hypercyclic operator T such that $T \oplus T$ is not hypercyclic if and only if there is an invertible hypercyclic operator $\tilde{T}$ such that $\tilde{T} \oplus \tilde{T}$ is not hypercyclic.*

Proof. If T is hypercyclic then the corresponding invertible operator $\tilde{T} : G \to G$ on its space of backward orbits G, is also hypercyclic. On the other hand, $G \oplus G$ is the space of backward orbits of $T \oplus T$ and $\tilde{T} \oplus \tilde{T}$ is the operator induced by $T \oplus T$. So, if the latest is not hypercyclic, we conclude that $\tilde{T} \oplus \tilde{T}$ is not hypercyclic. ■

We can finally establish the equivalence of all hypercyclicity criteria via the invertible operator $\tilde{T} : G \to G$.

Theorem 2.3 *All hypercyclicity criteria are equivalent.*

Proof. If $T : E \to E$ satisfies Bès criterion, then $T \oplus T$ is hypercyclic [5, Thm. 2.3], and so is $\tilde{T} \oplus \tilde{T}$ by the previous result. This implies that $\tilde{T}$ is hereditarily hypercyclic with respect to some increasing sequence (m_k) of integers [5, Thm. 2.3]. In particular we have that, for any $U, V \subset G$ non-empty open sets, there is $k \in \mathbb{N}$ such that

$$\tilde{T}^{m_k}(U) \cap V \neq \emptyset.$$

Equivalently, if B is the backward shift on G,

$$U \cap B^{m_k}(V) \neq \emptyset.$$

This yields the existence (see, e.g., [10, Thm. 1]) of $y \in G$ satisfying

$$\overline{\{B^{m_k} y \ : \ k \in \mathbb{N}\}}^G = G = \overline{\{\tilde{T}^{m_k} y \ : \ k \in \mathbb{N}\}}^G.$$

In other words, if $y = (y_1, y_2, \dots)$, then

$$(*) \quad \overline{\{(y_{m_k+1}, y_{m_k+2}, \dots) \ : \ k \in \mathbb{N}\}}^G = G = \overline{\{(T^{m_k} y_1, T^{m_k} y_2, \dots) \ : \ k \in \mathbb{N}\}}^G.$$

We define $S y_m := y_{m+1}$, $m \geq 1$. S is well defined since $y_m \neq y_n$ if $m \neq n$. Otherwise, $y_m = y_n$ for $m > n$ would mean $T^{m-n} y_n = y_n$, that is, $T^{m-n} y_1 = y_1$ and therefore y_1 would be a periodic point for T, contradicting the density of $\{(T^{m_k} y_1, \dots) \ : \ k \in \mathbb{N}\}$ in G by just looking at the first coordinate.

If we set $P_j : G \to E$ the j-th projection, $j \geq 0$, we get that

$$Y := \{y_m \ : \ m \geq 1\} = \{P_1\left(B^{m-1} y\right) \ : \ m \geq 1\}$$

is dense in E. Moreover, if we select a subsequence (j_k) of (m_k) such that $\lim_k B^{j_k} y = 0$, then

$$\lim_k S^{j_k} y_m = \lim_k y_{m+j_k} = \lim_k P_m\left(B^{j_k} y\right) = 0$$

for all $m \geq 1$. On the other hand, since $\tilde{T}$ is hereditarily hypercyclic with respect to (m_k), so is T, and we can find $x_0 \in E$ such that $\{T^{j_k} x_0 \ : \ k \in \mathbb{N}\}$ is dense in E. Pick then a subsequence (n_k) of (j_k) satisfying $\lim_k T^{n_k} x_0 = 0$ and define $X := \{T^m x_0 \ : \ m \geq 1\}$. We finally conclude

(i) $\lim_k T^{n_k} x = 0$, for all $x \in X$,

(ii) $\lim_k S^{n_k} y = 0$, for all $y \in Y$, and

(iii) $TSy = y$, for all $y \in Y$. ■

Observation: The following improvement of Gethner-Shapiro's hypercyclicity criterion can be obtained: The dense sets $X, Y \subset E$ can be supposed subspaces and $S : Y \to Y$ can be taken linear. Indeed, it is not difficult to notice that the sequence of vectors (y_m) in the previous proof is actually linearly independent. We then set $X := \operatorname{span} \operatorname{Orb}(T, x_0)$, $Y := \operatorname{span}(y_m)$, and extend S to Y by linearity. (i), (ii) and (iii) are also satisfied for these new sets. We do not know, however, whether the dense subsets X and Y can be supposed to be equal in general.

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