Bulletin de la Société Royale des Sciences de Liège, Vol. 69, 1, 2000, pp. 13-15

TIETZE - TYPE THEOREM FOR LOCALLY NONCONICAL CONVEX SETS

J.CEL

Abstract

A convex subset Q of a real topological linear space L is called *locally nonconical at a point* $q \in Q$ if and only if there exists a relative neighbourhood Q_q of q in Q such that for every two points $x, y \in Q_q$ there is a relative neighbourhood Q_x of x in Q such that $Q_x + \frac{1}{2}(y - x) \subseteq Q$. Q is called *locally nonconical* if and only if this condition is satisfied for every two points $x, y \in Q$. It is proved that Q is locally nonconical if and only if it is locally nonconical at every boundary point belonging to Q. This contributes to a recent work of Shell.

Key words: locally nonconical convex set, Tietze-type theorem.
1991 Math.Subj.Classif.: 46A55, 52A07.

Let Q be a convex subset of a real topological linear space L. Q is called *locally nonconical at a point* $q \in Q$ if and only if there exists a relative neighbourhood Q_q of q in Q such that for every two points $x, y \in Q_q$ there is a relative neighbourhood Q_x of x in Q such that $Q_x + \frac{1}{2}(y - x) \subseteq Q$. Q is called *locally nonconical* if and only if the above condition is satisfied for every two points $x, y \in Q$. A motivation for studying such sets has been given in [5] and [6].

Tietze's famous local characterization of convexity states in the general form that a closed connected subset S of L consisting exclusively of weak local convexity points is convex [7, Th.4.4]. In [1, Cor.2.3] the author obtained Tietze's theorem for an open connected subset S of L. Further generalizations of this theorem can be found in [2] and [4], and its variant for starshaped sets in [3]. The purpose of this note is to prove a new local versus global result for a recently introduced class of locally nonconical convex sets. Observe that the analogous theorem for the subclass of strictly convex sets is trivial.

Theorem. *A convex subset Q of a real topological linear space L is locally nonconical if and only if it is locally nonconical at every boundary point belonging to Q.*

Proof. Observe that, by [6, Remark 1.1], every open convex set in L is locally nonconical and it is easily seen that in this case the second part of the desired equivalence is vacuously true, so that we are done and assume in the sequel that $Q \cap \operatorname{bdry} Q \neq \emptyset$. The necessity of the condition is obvious, so that we prove its sufficiency. The following three claims will be useful.

Firstly, let three pairwise distinct points x_1, x_2, x_3 in Q be given. Suppose that there is a relative neighbourhood Q_1 of x_1 in Q such that $Q_1 + (x_2 - x_1) \subseteq Q$ and a relative neighbourhood Q_2 of x_2 in Q such that $Q_2 + (x_3 - x_2) \subseteq Q$. We claim that then there is a relative neighbourhood Q_0 of x_1 in Q such that $Q_0 + (x_3 - x_1) \subseteq Q$. By assumption, $Q_1 = Q \cap U_{x_1}$ and $Q_2 = Q \cap U_{x_2}$ for neighbourhoods U_{x_1} and U_{x_2} of points x_1 and x_2, respectively. Put $Q_0 = Q \cap U_{x_1} \cap (U_{x_2} + (x_1 - x_2))$ as the desired relative neighbourhood of x_1 in Q. In fact, on one hand $Q_0 + (x_3 - x_1) = (Q_0 + (x_2 - x_1)) + (x_3 - x_2) \subseteq U_{x_2} + (x_3 - x_2)$ and on the other hand $Q_0 + (x_3 - x_1) \subseteq (Q_1 + (x_2 - x_1)) + (x_3 - x_2) \subseteq Q + (x_3 - x_2)$, so that $Q_0 + (x_3 - x_1) \subseteq Q \cap U_{x_2} + (x_3 - x_2) = Q_2 + (x_3 - x_2) \subseteq Q$, as desired.

Secondly, select arbitrarily distinct points $x, y \in Q$ with the property that for every point $z \in [x, y]$ there is a relative neighbourhood Q_z of z in Q such that $Q_z + \frac{1}{2}(y - z) \subseteq Q$. We claim that then for every natural number n there exists a relative neighbourhood $Q_{x,n}$ of x in Q such that $Q_{x,n} + (1 - \frac{1}{2^n})(y - x) \subseteq Q$. We apply the induction on n. The case $n = 1$ is obvious, so that assume the truth for $n - 1 \geq 1$ and consider the case of n. Hence, $Q_{x,n-1} + (1 - \frac{1}{2^{n-1}})(y - x) \subseteq Q$ for a relative neighbourhood $Q_{x,n-1}$ of x in Q. Put $y_n = x + (1 - \frac{1}{2^n})(y - x)$. By assumption, there is a relative neighbourhood $Q_{y_{n-1}}$ of y_{n-1} in Q such that $Q \supseteq Q_{y_{n-1}} + \frac{1}{2}(y - y_{n-1}) = Q_{y_{n-1}} + (y_n - y_{n-1})$ and, by induction hypothesis, $Q_{x,n-1} + (y_{n-1} - x) \subseteq Q$. Consequently, the first claim above implies the existence of a relative neighbourhood $Q_{x,n}$ of x in Q for which $Q_{x,n} + (y_n - x) = Q_{x,n} + (1 - \frac{1}{2^n})(y - x) \subseteq Q$, as desired.

Thirdly, select arbitrarily distinct points $x, y \in Q$. Observe that since Q is convex, it must be locally nonconical at every interior point. Hence, Q is locally nonconical at every point of $[x, y]$. For every point $z \in [x, y]$ there exists, by [7, Th.1.4], a neighbourhood U_z of z in L starshaped relative to z and such that for every two points $r, s \in Q \cap U_z$ there is a relative neighbourhood Q_r of r in Q such that $Q_r + \frac{1}{2}(s - r) \subseteq Q$. Since $[x, y]$ is compact, it can be covered by finitely many such neighbourhoods. Denote by $V_1, \ldots, V_m$ line segments relatively open in $[x, y]$ being their intersections with $[x, y]$. Without loss of generality, assume that none of them is covered by a union of others and that they are so arranged that $V_1 \cap V_2 \neq \varnothing, V_2 \cap V_3 \neq \varnothing, \ldots, V_{m-1} \cap V_m \neq \varnothing$. We claim that for every $j = 1, \ldots, m$ there exist a point $v_j \in V_j$ and a relative neighbourhood $Q_{x,j}$ of x in Q such that $Q_{x,j} + (v_j - x) \subseteq Q$. We apply the induction on j. The case $j = 1$ is clear when we take $v_1 = x$. Suppose the truth for $j - 1 \geq 1$ and consider the case of j. Select any point $w \in V_{j-1} \cap V_j$ and, by virtue of induction hypothesis, a point v_{j-1} and a set $Q_{x,j-1}$ as described above. For nontriviality, let $v_{j-1} \notin V_j$. There is a natural number N such that $v_j = v_{j-1} + (1 - \frac{1}{2^N})(w - v_{j-1}) \in V_j$. By the second claim above for the line segment V_{j-1} there exists a relative neighbourhood $Q_{v_{j-1}}$ of v_{j-1} in Q such that $Q \supseteq Q_{v_{j-1}} + (1 - \frac{1}{2^N})(w - v_{j-1}) = Q_{v_{j-1}} + (v_j - v_{j-1})$. By induction hypothesis, there exists a relative neighbourhood $Q_{x,j-1}$ of x in Q such that $Q_{x,j-1} + (v_{j-1} - x) \subseteq Q$. The first claim above yields now a relative neighbourhood $Q_{x,j}$ of x in Q such that $Q_{x,j} + (v_j - x) \subseteq Q$, as desired.

To finish the proof select arbitrarily distinct points $x, y \in Q$ and assume that $\frac{1}{2}(x + y) \in V_k$ for some index $1 \leq k \leq m$. By the third claim above, there is a point $v_k \in V_k$ and a relative neighbourhood $Q_{x,k}$ of x in Q for which $Q_{x,k} + (v_k - x) \subseteq Q$. Select a natural number K such that $v_k' = \frac{2^{K-1}}{2^K - 1}(x + y) - \frac{1}{2^K - 1} v_k \in V_k$. It is seen that $\frac{x+y}{2} = v_k + (1 -$

$\frac{1}{2K})(v'_k - v_k)$, so that, by the second claim above, there is a relative neighbourhood Q_{v_k} of v_k in Q such that $Q \supseteq Q_{v_k} + (1 - \frac{1}{2K})(v'_k - v_k) = Q_{v_k} + (\frac{1}{2}(x+y) - v_k)$. But $Q_{x,k} + (v_k - x) \subseteq Q$, so that, by the first of above claims, there exists a relative neighbourhood Q_x of x in Q for which $Q_x + (\frac{1}{2}(x+y) - x) = Q_x + \frac{1}{2}(y - x) \subseteq Q$, as desired.

The proof is complete. □

Acknowledgment

Thanks are due to the Institute of Mathematics of the Pedagogical University of Kielce and the Circuit Theory Division of the Technical University of Łódź for the support during the preparation of this paper.

References

[1] Cel,J.: An optimal Krasnosel'skii-type theorem for an open starshaped set. *Geom. Dedicata* **66** (1997) 293-301.

[2] Cel,J.: A generalization of Tietze's theorem on local convexity for open sets. *Bull. Soc. Roy. Sci. Liège* **67** (1998) 31-33.

[3] Cel,J.: Characterizing starshaped sets by maximal visibility. *Geom. Dedicata* **74** (1999) 135-137.

[4] Sacksteder,R., Straus,E.G. and Valentine, F.A.: A generalization of a theorem of Tietze and Nakajima on local convexity. *J. London Math. Soc.* **36** (1961) 52-56.

[5] Shell,G.C.: Locally nonconical convex sets. PhD thesis, University of Michigan, Ann Arbor, 1997.

[6] Shell,G.C.: On the geometry of locally nonconical convex sets. *Geom. Dedicata* **75** (1999) 187-198.

[7] Valentine,F.A.: *Convex Sets*. McGraw-Hill, New York, 1964

J. Cel
Warszawska 24c/20
26-200 Końskie
Poland