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UNBIASED ESTIMATION OF THE SPECIFIC EULER-POINCARE CHARACTERISTIC

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ABSTRACT

The structures considered in the present paper are the realisation of a stationary ergodic random set and the problem is to estimate their specific Euler-Poincaré characteristic. The estimator proposed here leads to unbiased results whatever the space dimension. It has been tested on a 3D face centred cubic Bernoulli grid with densities ranging from 0 to 1. Moreover, the variance of this estimator has been calculated experimentally for every dimension and for all possible densities. With a large number of fields (~10000), these experimental results fit the theoretical curves calculated for 0, 1 and 2D spaces.

Key words: Euler-Poincaré characteristic, stereology, variance.

INTRODUCTION

The Euler-Poincaré characteristic (E.P.C.) or connectivity number, $N_n(X)$ measured in R^n space, provides a topological description of a set X. For a bounded set totally explored through successive fields, a method called the « shell-correction » had been proposed to evaluate the E.P.C. by Bhanu Prasad et al. (1989). This correction has also been applied by Gundersen et al. (1993) for biological studies.

This method has been naturally extended to stationary ergodic random sets (Jouannot, 1994), only for which specific values of the E.P.C. [Np, Nl, Na and Nv] are meaningful. These parameters are linked, via the classical stereological relationships (De Hoff and Rhines, 1968) (Serra, 1982), to the volume fraction of a phase [V_V] or the specific surface area [S_V] the integral of mean curvature [M_V] and the integral of Gaussian curvature [G_V] of the interface. Unfortunately, owing to edge effects, an estimator based on the « shell-correction » method is only asymptotically unbiased for digitized sets and it leads to misleading results for small field sizes (Jouannot and Jernot, 1993).

A new procedure of measurement leading to an unbiased estimation of the specific E.P.C. is then presented in the following section.

UNBIASED ESTIMATOR OF THE SPECIFIC E.P.C.

We consider a stationary ergodic digitized set X and a bounded field D belonging, for example, to R^1 space. By definition:

$$Nl\,(X) = \lim_{D \to R^1} \frac{N_1(X \cap D)}{|D|} \tag{1}$$

From the Euler formula,

$$N_1\,(X \cap D) = n_D({}^{v}1) - n_D({}^{v}1{-}1) \tag{2}$$

where $n_D({}^{v}1)$ stands for the number of points « ${}^{v}1$ » and $n_D({}^{v}1{-}1)$ the number of segments « ${}^{v}1{-}1$ » inside the field D.

Combining Eq. 1 and Eq. 2 leads to :

$$Nl\,(X) = P({}^{v}1) - P({}^{v}1{-}1) \tag{3}$$

where, for example,

$$P({}^{v}1{-}1) = \lim_{D \to R^1} \frac{n_D({}^{v}1{-}1)}{|D|} \tag{4}$$

is the probability of occurrence of the configuration «${}^{v}1{-}1$» inside the stationary random set. In order to obtain an unbiased estimate of Nl (X), unbiased estimates of $P({}^{v}1)$ and $P({}^{v}1{-}1)$ are thus required.

Generally speaking, the unbiased estimator of P(B), for the configuration « B », inside the field D

$$\hat{P}(B) = \frac{n_{D \ominus B}\,(B)}{|D \ominus B|} \tag{5}$$

completely solves the problem of the estimation of *specific* E.P.C.

All the configurations « B_i » to be tested lead to several different $|D \ominus B_i|$ field sizes. For the sake of simplification, all the probabilities $P(B_1)$, $P(B_2)$,..., $P(B_n)$ have been estimated as:

$$\hat{P}(B_i) = \frac{n_{D \ominus B_0}\,(B_i)}{|D \ominus B_0|} \tag{6}$$

where B_0 is a symmetrical structuring element containing all of the B_i 's.

For an image field of size $(e)^2$ - hexagonal grid, R^2 - or $(e)^3$ - f.c.c. grid, R^3 -, the measurement field is thus respectively $(l)^2 = (e-2)^2$ or $(l)^3 = (e-4)^3$. The practical measurements are illustrated for R^2 space on Fig. 1.

SPECIFIC E.P.C. OF STATIONARY ERGODIC RANDOM STRUCTURES

The proposed estimator has been tested on the realisation of a stationary ergodic random set. The structures are made up with points randomly placed on the nodes of a 3D face centred cubic Bernoulli grid, each node having the same chance to belong to the structure. Increasing the number of points, structures possessing densities ranging from 0 to 1 can be obtained. For each density, 10000 structures of size 34 x 34 x 34 are simulated and, according to the procedure described above, the specific E.P.C. is measured on a 30 x 30 x 30 reference size.

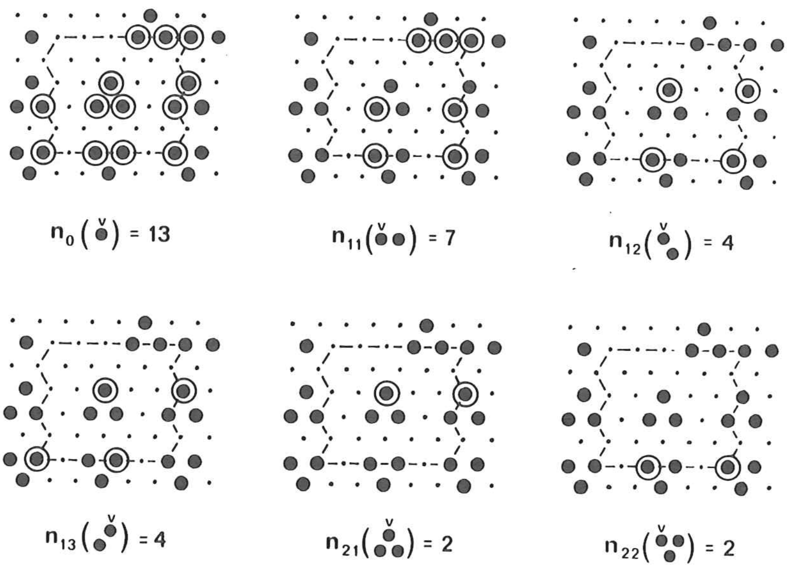

Fig. 1 : Measurement of the specific E.P.C. on a 8 x 8 binary image. The symbol « v » designates the origin of the configuration and each configuration is taken into account only if its origin belongs to the field of measurement (dotted line). The origin is then located by a circle.

$$\hat{Np} = n_0 / P = 13/(6.6)$$
$$\hat{Nl} = \{ n_0 - n_{11} \} / L = (13-7)/(6.6.a)$$
$$\hat{Na} = \{ n_0 - (n_{11} + n_{12} + n_{13}) + (n_{21} + n_{22}) \} / A = (13-15+4)/(6.a.6.a.\sqrt{3}/2)$$

where « a » is the real distance between two adjacent pixels.
Simplified expressions of the Euler formula $\hat{Nl} = n(\overset{v}{1}\ 0)/L$ and $\hat{Na} = [n(\overset{v}{{}_0{}^1{}_0}) - n(\overset{v}{{}^1{}_0{}^1})]/A$ give the same results $\hat{Nl} = 6/L$ and $\hat{Na} = (7-5)/A$.

MEAN VALUES

The mean values of the specific E.P.C. are reported as a function of the compacity in Fig. 2a, 3a, 4a and 5a. The results fit the theoretical curves already given in Jernot and Jouannot (1993) except that the area (R^2 space) and the volume (R^3 space) of the unit cell have been chosen here equal to 1 instead of $\sqrt{3}/2$ and $1/\sqrt{2}$. The misleading size effect previously observed (Jouannot and Jernot, 1993) completely disappears with the new procedure. Even with measurement fields of size 1, a good estimation is obtained as can be seen in Tbl. 1.

Table 1 : Experimental values (obtained from 100000 fields of size one) and theoretical ones.

Experimental				Theoretical			
Np	10^3 Nl	10^3 Na	10^3 Nv	Np	10^3 Nl	10^3 Na	10^3 Nv
0.1003	90.71	72.47	49.01	0.1	90	72	48.04
0.1982	157.55	95.24	23.13	0.2	160	96	23.81
0.2982	209.63	83.80	-28.80	0.3	210	84	-29.02
0.4005	239.67	47.23	-77.75	0.4	240	48	-75.65
0.5000	249.12	-0.10	-93.76	0.5	250	0	-93.75

VARIANCES

The variance of this estimator of the E.P.C. has been calculated in R^0, R^1, R^2 and R^3 spaces for each experimental density. The dispersion of the variance varying as the inverse of the number of the fields, a large number of fields (~ 10000) must be used. The results are presented in Fig. 2b, 3b, 4b and 5b for fields of size 30.

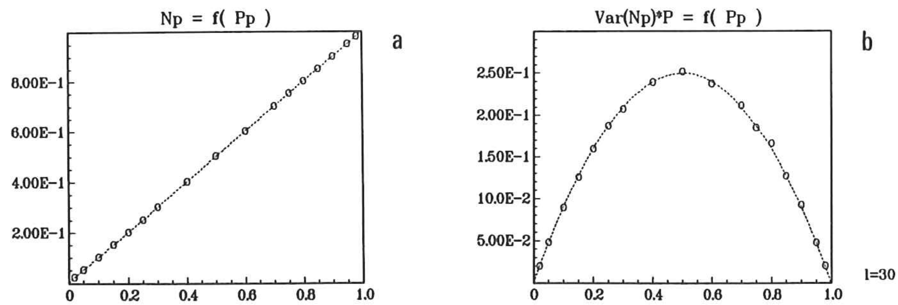

Fig. 2. (a) Specific E.P.C. in R^0 space, Np, as a function of the point fraction, Pp.
(b) Theoretical curve and experimental points for associated normalized variance (P=30.30.30).

For a stationary ergodic random function with finite and non-zero integral range, the variance evolves as the inverse of the field size (Lantuéjoul, 1991). The experimental values of the variance have then been multiplied by the sizes of the measurement fields and are called « normalized variance », allowing a direct comparison between different sizes.

The theoretical curves for the Bernoulli grid have also been calculated in R^0, R^1 and R^2 spaces. Let us explain the principle of the calculations for R^1 space.

We assume that the measurements can be made inside a field Dl of size l. Nl is estimated from : $\hat{Nl}(X) = n_{Dl}\,({}^{v}1\ 0)\,/\,l$.

The random function (Z_x, $x \in Dl$) is introduced :

Zx = 1 if the configuration $({}^{v}1\ 0)$ appears at the point x,

Zx = 0 otherwise.

The estimator of Nl is then rewritten as : $\hat{Nl}(X) = \frac{1}{l} \sum_{x \in Dl} Zx$.

This estimator, leading to the theoretical formula, is unbiased :

$E\{\hat{Nl}(X)\} = \frac{1}{l} \sum_{x \in Dl} E\{Zx\} = E\{Zx\} = p(1-p)$ where p is the concentration of points.

Then, the associated variance is equal to : $\mathrm{Var}\{\hat{\mathrm{Nl}}(\mathrm{X})\} = E\{\hat{\mathrm{Nl}}(\mathrm{X})^2\} - [\,E\{\hat{\mathrm{Nl}}(\mathrm{X})\}\,]^2$

with $E\{\hat{\mathrm{Nl}}(\mathrm{X})^2\} = \frac{1}{l^2} \sum_{\mathrm{x,y} \in \mathrm{D}l} E\{\mathrm{Zx.Zy}\}$.

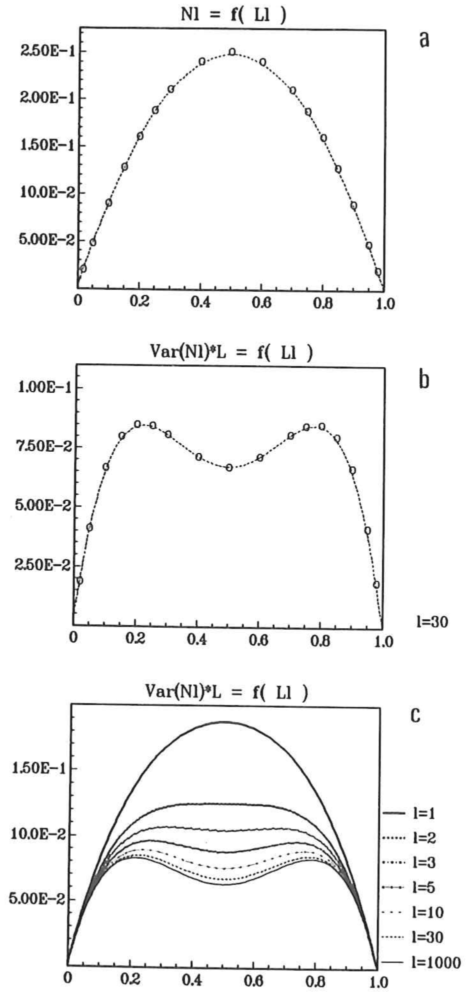

Fig. 3 :

(a) *Specific E.P.C. in R^1 space, Nl, as a function of the lineal fraction Ll.*

(b) *Theoretical curve and experimental points for associated normalized variance (L=30).*

(c) *Theoretical variances for different sizes.*

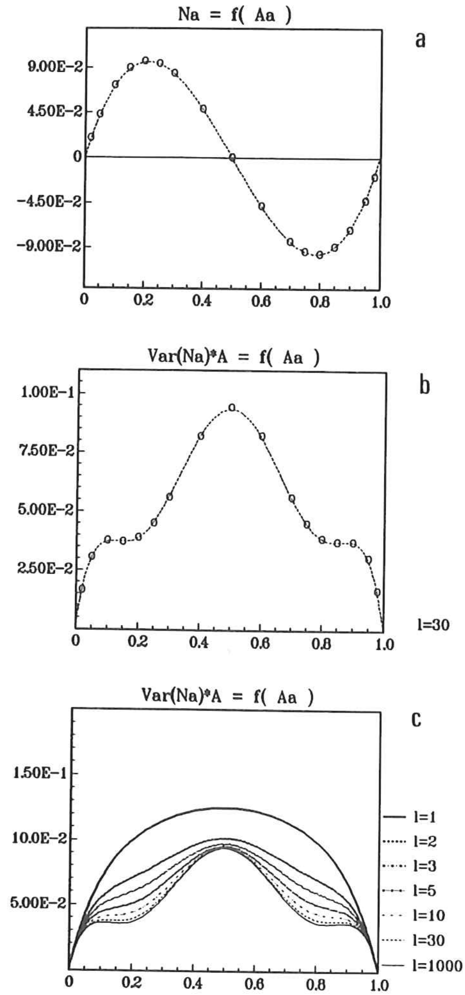

Fig. 4 :

(a) *Specific E.P.C. in R^2 space, Na, as a function of the area fraction Aa.*

(b) *Theoretical curve and experimental points for associated normalized variance (A=30.30).*

(c) *Theoretical variances for different sizes.*

The value of $E\{Zx.Zy\}$ depends on the relative positions x and y :

if $|y - x| = 0$ $\quad E\{Zx.Zy\} = E\{Zx\} = p(1\text{-}p)$

if $|y - x| = 1$ $\quad E\{Zx.Zy\} = 0$

if $|y - x| > 1$ $\quad E\{Zx.Zy\} = E\{Zx\}.\,E\{Zy\} = [\,p(1\text{-}p)\,]^2$

This leads to : $\operatorname{Var}\{\hat{Nl}(X)\} = \dfrac{1}{l^2}\Big[\, l p(1-p) + 0 + (l^2 - 3l + 2)\,[\,p(1-p)\,]^2 \,\Big] - [\,p(1-p)\,]^2$

The same calculations have been performed for R^0 and R^2 using $({}_0\overset{v}{{}^1}{}_0)$ and $({}^1\overset{v}{{}_0}{}^1)$ in the case of two-dimensional space (hexagonal grid). Using $\alpha = p(1-p)$, we obtain :

$$\operatorname{Var}(\hat{Np}(X)) \times P = \alpha\,[\,1\,] \tag{7}$$

$$\operatorname{Var}(\hat{Nl}(X)) \times L = \alpha\,[\,1 - 3\alpha + \frac{1}{l}\,(2\alpha)\,] \tag{8}$$

$$\operatorname{Var}(\hat{Na}(X)) \times A = \alpha\,[\,1 - 9\alpha + 26\alpha^2 + \frac{1}{l}\,(8\alpha - 32\alpha^2) + \frac{1}{l^2}\,(-2\alpha + 10\alpha^2)\,] \tag{9}$$

In Eq. 8 and Eq. 9, the terms depending on $(1/l)$ and $(1/l^2)$ are due to edge effects appearing only for small field sizes as can be seen on Fig. 3c and 4c. For $l = 30$, these effects are neglectible.

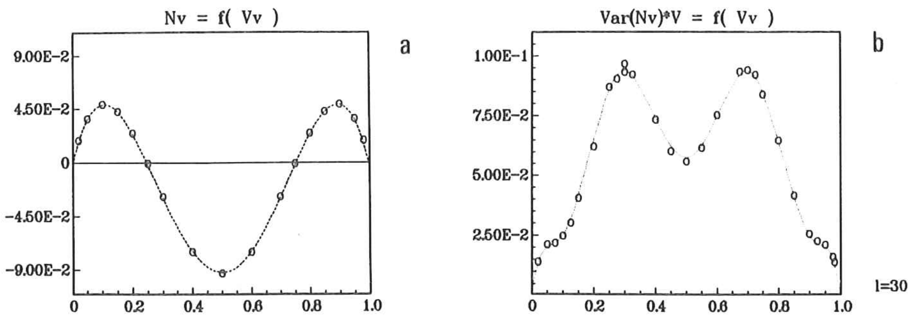

Fig. 5. (a) Specific E.P.C. in R^3 space, Nv, as a function of the volume fraction, Vv.
(b) Experimental points for associated normalized variance (V=30.30.30)

DISCUSSION

In two-dimensional space, the decomposition used for the simplified Euler formula (cf. Fig. 1), i.e. the difference between $({}_0\overset{v}{{}^1}{}_0)$ and $({}^1\overset{v}{{}_0}{}^1)$, corresponds exactly to the tangent count method applied on a digitized structure. This establishes a link between the classical stereology and the mathematical morphology based on structuring elements.

In continuous three-dimensional space, according to De Hoff (1987), the « volume tangent count » allows an unbiased estimation of the E.P.C. from two closely spaced planes (unfortunately, its practical application seems questionable). The same result is observed here for a digitized structure : identical mean values are obtained on measurement fields of size 30 x 30 x 30 and 165 x 165 x 1 starting from 32 x 32 x 31 and 167 x 167 x 2 images.

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