ACTA STEREOL 1996; 15/1: 3-8
ORIGINAL SCIENTIFIC PAPER

ANISOTROPIC STEREOLOGY OF FIBRES AND SURFACES

Viktor Beneš, Joël Chadoeuf[1], Joachim Ohser[2]

Dept. of Mathematics FSI, Czech Technical University, Karlovo nám.13, 12135 Prague 2, Czech Republic
[1]Biométrie, INRA, Domaine St Paul, BP 91, 84140 Montfavet, France
[2]Institute of Ferrous Metallurgy, Technical University of Mining and Technology, D-09596 Freiberg, Germany

ABSTRACT

The paper is devoted to the stereological estimation of length and surface area in anisotropic structures. Statistical properties of various estimators are investigated. The relation between estimators based on projections and intersections is studied. Finally an implementation of some estimators in image analysis is described. The paper is partly a review of recent results and is presented in a condensed way. The details including mathematical proofs can be found in the referenced papers.

Keywords: anisotropic fibres and surfaces, intensity estimation, projection measure, variance.

INTRODUCTION

The paper presents a review of results concerning the statistical properties investigation of intensity estimators of stationary anisotropic random fibre and surface processes in R^d. Basically, four sampling schemes are included for the estimation as classified in the following table.

Table 1. Classification of sampling schemes for fibre and surface processes.

	direct probes	indirect probes
fibres	total projection on R^1	total projection on R^{d-1}
surfaces	total projection on R^{d-1}	intersection with hyperplane

By projection we always mean the total projection, i.e. measuring multiplicities of objects overlapping by projection. Direct probes are followed by an immediate measurement while indirect probes transform the process only and new probes should be applied after it.
In the first two sections various known estimators are defined by means of the projection measure and their unbiasedness, consistency and efficiency are studied by methods developed in Ohser(1991), Beneš et al.(1994), Beneš(1995), Beneš at al.(1995). Explicit formulas for variances are obtained for Boolean models (Krejčíř and Beneš, 1995).
In practical stereological work the total projection is approximated by means of counting the intersection number between the structure and probe. In section 3 this situation is described in two ways (Chadoeuf and Beneš, 1994). First the difference between variances of estimators based on projection and intersection measure can be evaluated. Secondly the variance of the

difference between the observed quantity of the structure and its intersection with the probe is evaluated.
In the final section an application of the projection estimator in the plane in image analysis is described (Ohser, 1995). It is of great practical importance because of the special type of observable orientations by an image analyser.

DIRECT PROBES

Let Φ be a stationary random fibre (surface) process on the Euclidean space $(R, \mathcal{B}, \nu)^d$ with Borel σ-algebra and Lebesgue measure. Let λ be the intensity constant of Φ and $\mathcal{R}$ its rose of directions (a probability measure of the space $(M, \mathcal{M})^d$ of axial orientations). For a bounded window $B \in \mathcal{B}^d$ with $\nu(B) > 0$ let $\Phi(B)$ be the length (surface area) of Φ in B. Let Q be a probability measure on $\mathcal{M}$ interpreted as the distribution of probe orientations (sampling design). The projection measure Φ_Q on $\mathcal{B}$ is defined by

$$\Phi_Q(B) = \int_B \mathcal{F}_Q(m(x))\Phi(dx), \quad B \in \mathcal{B}, \tag{1}$$

where $\mathcal{F}_Q(l) = \int_{M^d} \mid \cos \sphericalangle(l, m) \mid Q(dm)$, $l \in M^d$, the cosine transform of Q, is interpreted as the measure of total projection of Φ in B averaged with respect to Q. The intensity constant λ_Q of Φ_Q is equal to $\lambda_Q = \lambda \mathcal{F}_{\mathcal{R}Q}$, where $\mathcal{F}_{\mathcal{R}Q} = \int_{M^d} \mathcal{F}_{\mathcal{R}}(l)Q(dl)$. Then the basic unbiased estimator $\hat{\lambda}$ based on direct probes is

$$\hat{\lambda} = \frac{\Phi_Q(B)}{\nu(B)\mathcal{F}_{\mathcal{R}Q}}. \tag{2}$$

For the variance of this estimator it holds that

$$var\hat{\lambda} = \left(\frac{\lambda}{\nu(B)}\right)^2 \int_{R^d} g_B(x)(p_Q(x) - 1)dx, \tag{3}$$

where $g_B(x) = \nu(B \cap B_{-x})$ is the set covariance function. We denote by p, p_Q the pair correlation function (pcf) of Φ, Φ_Q, respectively. They are related by the formula $p_Q(x) = \frac{I_Q(x)}{\mathcal{F}^2_{\mathcal{R}Q}} p(x)$, where $I_Q(x) = \int\int \mathcal{F}_Q(m)\mathcal{F}_Q(l)\mathcal{W}_x(d(m,l))$, $\mathcal{W}_x$ being the two-point orientation distribution of Φ. Explicit formulas for the variance of (2) are available in the case of Boolean models.

Theorem 1 *Let Φ be a Boolean model of compact subsets of straight lines (hyperplanes), then*

$$var\Phi_Q(B) = \lambda \int_M \mathcal{F}_Q^2(m) \int g_B(x) f_m(dx)\mathcal{R}(dm), \tag{4}$$

where $f_m(K)$, $K \in \mathcal{B}^d$ is the mean length (surface area) of $S \cap K$ of the fibre (surface) $S \subset \Phi$ hitting the origin with orientation m.

Simpler formulas are obtained for B a ball since then g_B does not depend on orientations.
An equivalent definition of the projection measure is $\Phi_Q(B) = \int_M \int_{Proj_l(B)} N_y dy Q(dl)$, where N_y is the number of intersection points of $\Phi \cap B$ with test hyperplane (line) located at $y \in R^1$ $(y \in R^{d-1})$, $Proj_l(B)$ is the projection of B onto line l (in direction l.) In practice the inner integral is approximated by a sum corresponding to a finite number of test probes. So the estimator (2) is a continuous approximation of various classical estimators according to the choice of sampling design Q :
a) $Q = \mathcal{U}$ uniform distribution, then (2) is equivalent to the estimator $\hat{\lambda} = \frac{\Phi(B)}{\nu(B)}$;
b) $Q = \delta_l$ denotes projection on (in) a single direction $l \in M$, estimator is dependent on $\mathcal{R}$;
c) spatial grid estimator (Sandau, 1987) $Q = \frac{1}{3}(\delta_x + \delta_y + \delta_z)$, $x, y, z \in M^3$ mutually perpendicular. Estimator $\hat{\lambda} = \frac{2\Phi_Q(B)}{\nu(B)}$ is unbiased for $\mathcal{R} = \mathcal{U}$ only.

d) vertical spatial grid (Cruz-Orive and Howard, 1994). Let $m = (\theta, \varphi) \in M^3$, θ, φ being the colatitude, longitude, respectively. Put $Q_\xi(d(\theta,\varphi)) = \frac{1}{4}(\delta(\xi - \varphi) + \delta(\xi + \frac{\pi}{2} - \varphi)) \sin\theta d\theta d\varphi$, let ξ has uniform distribution on $\langle 0, \pi)$. This sampling randomization makes the unbiased estimator $\hat{\lambda} = \frac{2\Phi_{Q_\xi}(B)}{\nu(B)}$ independent of $\mathcal{R}$.
Variances of estimators a)-d) were evaluated in Beneš(1995) and explicitly compared for Poisson processes with Dimroth-Watson distribution $\mathcal{R}$ on $\mathcal{M}^3$. In Baddeley and Cruz-Orive(1995) the non-existence of minimum variance unbiased estimator in stereological applications is emphasized due to the "incompleteness" of observable information. The authors present counterexamples when lower dimensional sampling probes yield smaller estimation variances. We obtain further natural examples of this phenomenon in anisotropic structures, see the final example in Beneš et al.(1995).

INDIRECT PROBES

Let $\alpha \in M^d$ be the orientation of an indirect probe (projection foil of thickness t, section hyperplane for a fibre or surface process Φ). The transformed process Φ^α in R^{d-1} has a rose of directions $\mathcal{R}_\alpha$ on $\mathcal{M}^{d-1}$. Denote $J_{Q\mathcal{R}}^{\alpha n} = \mathcal{G}_\mathcal{R}(\alpha) \int \mathcal{F}_Q^n(l)\mathcal{R}_\alpha(dl)$ for an integer exponent n, $\mathcal{G}_\mathcal{R}(l) = \int_M \sin \sphericalangle(l,m)\,\mathcal{R}(dm)$, $l \in M$, being the sine transform of $\mathcal{R}$, Q the sampling design in M^{d-1}. Then the unbiased estimators $\hat{S}$, $\hat{L}$ of surface, length intensity λ, respectively, are

$$\hat{S} = \frac{\Phi_Q^\alpha(B)}{\nu(B) J_{Q\mathcal{R}}^{\alpha 1}} \quad \text{and} \quad \hat{L} = \frac{\Phi_Q^\alpha(B)}{t\nu(B) J_{Q\mathcal{R}}^{\alpha 1}}, \tag{5}$$

$B \in \mathcal{B}^{d-1}$ being a compact window of positive $\nu(B)$. The general theory starting with (3) can be applied to derive the estimation variances. Specially for the Poisson hyperplane process and a ball B, denoting $\mathcal{O}_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, it holds (Beneš, 1995) that

$$var\hat{S} = \mathcal{O}_{d-2}\lambda \frac{\int_0^\infty r^{d-3} g_B(r)dr}{[\nu(B)]^2} \frac{J_{Q\mathcal{R}}^{\alpha 2}}{(J_{Q\mathcal{R}}^{\alpha 1})^2}. \tag{6}$$

Projections of line processes on R^{d-1} lead to segment processes with generally dependent segment length and orientation. This more complicated situation is discussed in Beneš et al.(1995).
Again the estimators (5) depend on $\mathcal{R}$, which is in practice unknown. Two natural sampling randomizations are used to overcome this problem: IUR (isotropic uniform random) probes or VUR (vertical uniform random) sections (Baddeley, 1985) and projections (Gokhale, 1990). IUR, VUR sampling yields unbiased estimators

$$\hat{S} = \frac{\Phi^\alpha(B)}{\mu_d \nu(B)}, \quad \hat{L} = \frac{\Phi^\alpha(B)}{t\mu_d \nu(B)}, \tag{7}$$

$$\hat{S} = \frac{\Phi_Q^\xi(B)}{\omega_d \nu(B)}, \quad \hat{L} = \frac{\Phi_Q^\xi(B)}{t\omega_d \nu(B)}, \tag{8}$$

respectively. Here α is uniform random on M^{d-1}, $\omega_d = \mathcal{F}_\mathcal{U}(l), \mu_d = \mathcal{G}_\mathcal{U}(l)$ constants for arbitrary $l \in M^d$. In polar coordinate system $l = (l_1, ..., l_{d-1})$ let $\xi = (\frac{\pi}{2}, \frac{\pi}{2}, ...\frac{\pi}{2}, l_{d-1})$, where l_{d-1} is uniform random. Then Q in (8) is a fixed measure on $\mathcal{M}^{d-1}$: $Q(dl_1...dl_{d-2}) = \frac{2\pi}{\mathcal{O}_d} \sin^{d-2} l_1 ... \sin l_{d-2} dl_1...dl_{d-2}$.
Estimation variances in randomized sampling $var\hat{\lambda} = var(E(\hat{\lambda}|\alpha)) + E(var(\hat{\lambda}|\alpha))$ have two terms. For IUR sampling in (8) we have e.g. $var(E(\hat{\lambda})) = \frac{\lambda^2}{\mu_d^2} var\mathcal{G}_\mathcal{R}(\alpha)$ for both $\hat{S}$ and $\hat{L}$, this term causes that the estimators are inconsistent. Explicit formulas for the second term are based on the general theory.

For the Poisson line process Φ and a ball B we have e.g. for IUR sampling

$$var(\hat{L}|\alpha) = \frac{\lambda}{t} \frac{\mathcal{G}_{\mathcal{R}}(\alpha)}{[\mu_d \nu(B)]^2} \int_M \int_0^\infty g_B(r) df_m(r) \mathcal{R}_\alpha(dm)$$

and for VUR sampling

$$var(\hat{L}|\xi) = \frac{\lambda}{t} \frac{\mathcal{G}_{\mathcal{R}}(\xi)}{[\omega_d \nu(B)]^2} \int_M \mathcal{F}_Q^2(m) \int_0^\infty g_B(r) df_m(r) \mathcal{R}_\xi(dm).$$

Comparison of estimation variances in R^3 for Dimroth-Watson distribution $\mathcal{R}$ is presented in Beneš(1995).

PROJECTIONS AND INTERSECTIONS

The important step from non-realizable continuous sampling to practical discrete sampling was studied in Chadoeuf and Beneš(1994) in R^2. Let Φ be a stationary fibre process, which is projected on $l \in M^2$ to obtain $\Phi_l = \Phi_{\delta_l}$. Let a rectangular window $B = \langle 0, a\rangle \times \langle 0, b\rangle$ have sides a, b parallel with x, y axis, respectively. Let the direction l coincide with the y-axis. N_y for given y denotes the number of intersections $\Phi \cap B$. For $h = \frac{a}{n}$, n integer, call $L_h(B) = h \sum_{i=1}^n N_{ih}$ the intersection measure of Φ in B. An unbiased intensity estimator

$$\hat{\lambda} = \frac{L_h(B)}{ab\mathcal{F}_{\mathcal{R}}(l)} \tag{9}$$

will be compared with (2). The reasoning is based on a model for the pair correlation function (pcf) p_l of Φ_l. In polar coordinates (r, θ), where $r > 0$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is the angle with y-axis. The pcf p_l is symmetric, i.e. it holds $p_l(r, \theta) = p_l(r, \theta + \pi)$. The following assumptions will be combined:
I) let p_l satisfy $p_l(r, \theta) = 1 + \frac{c(\theta)}{r} + h(r, \theta)$ for functions c, h continuous in $R^2 - \{0\}$, where $h(r, \theta) = g(\theta)(1 + o(1))$ for $r \to 0$.
II) function $c(\theta)$ satisfies $c(\theta) \sim K_+(\frac{\pi}{2} - \theta)^{\alpha_+}$ for $\theta \to \frac{\pi}{2}$, and $c(\theta) \sim K_-(\frac{\pi}{2} + \theta)^{\alpha_-}$ for $\theta \to -\frac{\pi}{2}$ and real constants $K_+, K_-, \alpha_+ > 0, \alpha_- > 0$.
III) function $g(\theta)$ satisfies $g(\theta) \sim D_+(\frac{\pi}{2} - \theta)^{\beta_+}$ for $\theta \to \frac{\pi}{2}$, and $g(\theta) \sim D_-(\frac{\pi}{2} + \theta)^{\beta_-}$ for $\theta \to -\frac{\pi}{2}$ and real constants $D_+, D_-, \beta_+ > 0, \beta_- > 0$.
The variance of estimator (9) is

$$var\hat{\lambda} = \frac{\sum_i \sum_j cov(N_{ih}, N_{jh})}{n^2 b^2 \mathcal{F}_{\mathcal{R}}^2(l)}.$$

Theorem 2 *Under the assumptions I-II in rectangular coordinates $p_u(t, y)$ it holds that*

$$cov(N_y, N_z) = \lambda^2 \mathcal{F}_{\mathcal{R}}^2(l) \int_{-b}^{b} (b - |t|)(p_l(t, y - z) - 1) dt \tag{10}$$

for $y \neq z$ and

$$varN_y = \lambda^2 \mathcal{F}_{\mathcal{R}}^2(l)(b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{c(\theta) d\theta}{\cos\theta} + \frac{b^2}{2}(g(\frac{\pi}{2}) + g(-\frac{\pi}{2}))).$$

Theorem 2 can be used for the comparison of estimation variances in (2) and (9). On the other hand the following result enables us to evaluate the asymptotic variance of the difference between the projection and intersection measure.

Theorem 3 *Under the conditions I-II for $h \to 0$ it holds that $L_h(B) \to \Phi_l(B)$ in quadratic mean. If moreover III holds with $\alpha_+, \alpha_-, \beta_+, \beta_-$ all greater than 1, then the speed of convergence is given by*

$$E((L_h(B) - \Phi_l(B))^2) = -\frac{ah^2}{6}\lambda^2 \mathcal{F}_{\mathcal{R}}^2(l)(b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{g(\theta)d\theta}{\cos^2\theta} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|\sin\theta| c(\theta)d\theta}{\cos^2\theta}) + o(h^2). \tag{11}$$

The proof in Chadoeuf and Beneš(1994) makes use of Matheron's(1971) result neglecting the "Zitterbewegung".

Example: Let Φ be an anisotropic Boolean segment process in R^2 with length intensity λ, rose of direction $\mathcal{R}$ independent of the segment length distribution function $H(r) = 1 - e^{-r/q}$, $r, q > 0$. Using the technique of Beneš et al.(1994) we get

$$p_l(r,\theta) = 1 + \frac{\rho(\theta)\cos^2\theta e^{-r/q}}{\lambda r \mathcal{F}_{\mathcal{R}}^2(l)},$$

where ρ is the probability density of $\mathcal{R}$. This is in fact model I) with $c(\theta) = \frac{\rho(\theta)\cos^2\theta}{\lambda \mathcal{F}_{\mathcal{R}}^2(l)}$ and $g(\theta) = -\frac{\rho(\theta)\cos^2\theta}{\lambda q \mathcal{F}_{\mathcal{R}}^2(l)}$. Let ρ satisfy $\lim_{\theta\to\pm\pi/2} \rho(\theta)(\theta \pm \frac{\pi}{2})^{1-\varepsilon} < \infty$ for some $\varepsilon > 0$. Then we have $\alpha_+, \alpha_-, \beta_+, \beta_-$ all greater than 1 in II), III) and from Theorem 2 $var N_y = b\lambda\mathcal{F}_{\mathcal{R}}(0)$,

$$cov(N_0, N_y) = 2y^2\lambda \int_0^b (b-t)\exp(-\frac{\sqrt{t^2+y^2}}{q})\frac{\rho(\arctan t/y)}{(t^2+y^2)^{3/2}}dt.$$

Finally using Theorem 3 we obtain $E(L_h(B) - \Phi_l(B))^2 = \frac{ah^2\lambda}{6}(\frac{b}{q} + \mathcal{G}_{\mathcal{R}}(0))$.

The generalization of these results to R^d is not straightforward as in sections 1,2 where the integral of pcf was to be evaluated, while here pcf itself is desirable. For some results in R^3 see Chadoeuf(1995).

APPLICATION IN IMAGE ANALYSIS

We consider a stationary random closed set Θ in R^2 and denote by Φ the fibre process of its boundary. In this situation the outer boundary normal orientation distribution $\mathcal{R}$ is studied on $\langle 0, 2\pi)$. Consider the length intensity estimator (2) in the form $\hat{\lambda} = \frac{\Phi(B)}{\nu(B)}$, $B \in \mathcal{B}^2$, where $\Phi(B) = \frac{1}{2}\int_0^{2\pi} \Phi_l(B)dl$. An alternative formula (Ohser, 1995)

$$\Phi_l(B) = \lim_{r\to 0}\int_B \frac{f_\Theta(x,r,l)dx}{r} \tag{12}$$

is used, where $f_\Theta(x,r,l) = 1_\Theta(x)[1 - 1_\Theta(x + (r,l))]$. To obtain a discrete approximation of this limit first consider a binary picture $C = (c_{ij})$, $c_{ij} = 1_\Theta(x_{ij})$, $i,j \in Z$ in a bounded window B. Here $x_{ij} = (i\triangle, jf\triangle)$ are grid points (pixels), $\triangle$ the digital resolution in the x-direction and f the aspect ratio, i.e. $f\triangle$ is the digital resolution in the y-direction. Using special digital filter $F_n = (f_{km})_{n\times n}$ of order n with $f_{km} = 2^{m+kn}$ and filtering C with F_n we get a grey-tone image $G = C * F_n$ with n^2 bits per pixel, $G = g_{ij}$, $g_{ij} = \sum_k\sum_m c_{i+k,j+m}f_{km}$, $i,j \in Z$. Now denoting $h_k = \sum_i\sum_j 1_W(x_{ij})(g_{ij} = k)$, $k = 0,...,N-1$, $N = 2^{n\times n}$ the grey-tone histogram of G, we obtain the discrete approximation of formula (12) in the direction l_i, $i = 0,..,8(n-1)-1$ as (Ohser, 1995)

$$\Phi_{l_i}(B) = d_i\sum_{k=0}^{N-1} h_k(k = k|m_i^{(0)})[1 - (k = k|m_i^{(1)})]. \tag{13}$$

Here d_i is the row distance in i-th direction, $m_i^{(\cdot)}$ are the masks (pairs of elements of F_n defining the direction l_i) and $(k = k|m) = 1$ for $k = k|m$ while $(k = k|m) = 0$ for $k \neq k|m$, where $k|m$ denotes the bit-wise *or* of the integers k and m. The intensity estimator has thus a form

$$\hat{\lambda}\nu(B) = \pi \int_0^{2\pi} \Phi_l(B) Q_n(dl) = \frac{1}{16(n-1)} \sum_{i=0}^{8(n-1)-1} (l_{i+1} - l_i)\Phi_{l_i}(B), \tag{14}$$

Table 2. Estimation variance

Order	$(J_n - 1)10^6$	
n	$f = 1$	$f = 2$
2	544	10366
3	137	7620
5	93	4384
...	...	...
∞	0	0

where Q_n is a probability measure on $\langle 0, 2\pi)$ corresponding to F_n. This estimator is of type (2), so the general theory yields its statistical properties. It is asymptotically unbiased when $n \to \infty$, its bias depends on the rose of direction $\mathcal{R}$. The variance follows from formulas (3) and (4). Let f_m in (4) be independent of m and B a circle in R^2. We study the variance factor $\int \mathcal{F}_{Q_n}^2(m)\mathcal{R}(dm)$ for $n \to \infty$. Assume that $\mathcal{R} = \mathcal{U}$ and denote $J_n = \frac{\pi}{2}\int_0^{2\pi} \mathcal{F}_{Q_n}^2(m)dm$, the asymptotics is demonstrated by Table 2 for different aspect ratio f.

ACKNOWLEDGEMENT

The research work of the first author was supported by the Grant Agency of the Czech Republic, project no. 201/93/2172.

REFERENCES

Baddeley A. An anisotropic sampling design. Geobild'85(ed. by Nagel W.), Friedrich-Schiller Universität Jena; 1985: 92-97.

Baddeley A, Cruz-Orive LM. The Rao-Blackwell theorem in stereology and some counterexamples. Adv Appl Probab 1995; 27: 2-19.

Beneš V. On second order formulas in anisotropic stereology. Adv Appl Probab 1995; 27: 326-43.

Beneš V, Chadoeuf J, Ohser J. On some characteristics of anisotropic fibre processes. Math Nachr 1994; 169: 5-17.

Beneš V, Krejčíř P, Ohser J. Projection measures and estimation variances of intensities. Statistics 1995; submitted.

Chadoeuf J, Beneš V. On some estimation variances in spatial statistics. Kybernetika 1994; 30: 245-262.

Chadoeuf J. Convergence of projection estimators in R^3. Techn Rep; INRA Montfavet 1995.

Cruz-Orive LM, Howard CV. Estimation of individual feature surface area with the vertical spatial grid. Report 321/LC 56, Universitat Bern 1994.

Gokhale AM. Unbiased estimation of curve length in 3D using vertical slices. J Microsc 1990; 159: 133-141.

Krejčíř P, Beneš V. Orientation analysis in second order stereology. Proc 9ICS Copenhagen 1995, Acta Stereol; in press.

Ohser J. Variances of different intensity estimators of the specific line length. Stoch Geom, Geom Statist, Stereol, Oberwolfach 1991; unpublished.

Ohser J. Random Minkowski measures and digitalization. Proc ICIAM Congress, Hamburg 1995; J Microsc, submitted.

Sandau K. How to estimate the area of a surface using a spatial grid. Acta Stereol 1987; 6/III: 31-36.

Stoyan D, Kendall WS, Mecke J. Stochastic Geometry and Its Applications. Akademie Verlag; Berlin 1987.

Presented at the 9th International Congress for Stereology, Copenhagen, August 20th to 25th, 1995.