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INTRODUCTION TO THE USE OF HOUGH TRANSFORM IN SHAPE ANALYSIS

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ABSTRACT

This paper shows, in an introductory manner, how Hough Transform can provide a versatile approach to many questions concerning planar convex shapes, with a special attention paid to origin changes. Applications are given in the domains of shape symmetry parameters and length computation. Radon Transform is shown to be a natural extension of Hough Transform to gray-level and/or non convex shapes.

Keywords :
Convexity, Hough transform, image processing, Radon transform, shapes, shape parameters, support functions.

INTRODUCTION

The word « shape » will hereafter designate the interior of a simple (not self intersecting) closed curve in the plane. A shape is considered as a mathematical model of a planar view of an object.

This introductory paper shows how interesting it can be to place shape study into the framework of Hough Transform (H.T.). It establishes the interaction between geometrical and analytical concepts, such as diameter and support functions, origin translation, barycentrical method, etc. Two significant applications are given : one in perimeter computation, the other one in shape parameters. A last part is devoted to Radon Transform, which extends the properties already met with H.T.

THE SUPPORT FUNCTION OF A CONVEX SHAPE

Apparent diameter and support function of a shape

A convex shape being given, let D_θ be **any** line with polar angle θ. Let $d(\theta)$ be the **apparent diameter function** (see figure 1), i.e., the length of the shape's projection on D_θ.

The main interest of function $d(\theta)$ is Crofton's theorem, which asserts that the mean value of $d(\theta)$ on $[0, \pi)$, multiplied by constant π, is equal to the shape's perimeter.

Function $d(\theta)$, though it is defined «in a natural manner» and is origin-independent, does not characterize, by itself, a unique shape : two different shapes, for example a disk and a « wheel », may possess the same diameter function (Fillère, 1995).

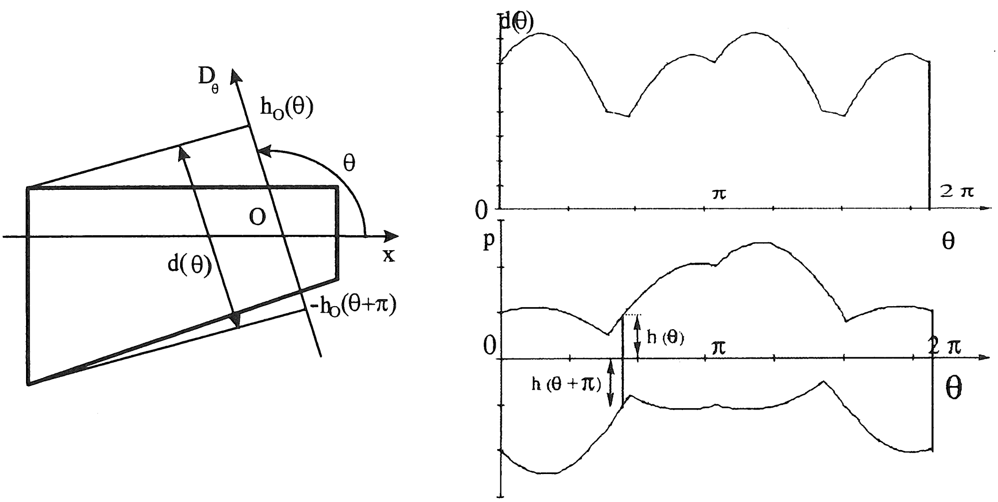


Fig. 1 : A shape, its associated functions $d(\theta)$ (diameter), $h_O(\theta)$ *(support) and* $-h_O(\theta+\pi)$.

A more fundamental origin-dependent function, which is associated in a unique way to a convex shape, is support function.

Here is a way to define it. Let us assume that a particular point O **interior to the shape** has been selected. Let D_θ be the oriented line with polar angle θ passing through O. Let us consider the endpoints of the shape's projection on D_θ (see figure 1). One of them has a positive abscissa on D_θ ; we will denote it $h_O(\theta)$ and will call $p = h_O(\theta)$ the **support function**. It is clear, using a π rotation, that the abscissa of the other endpoint is $p = - h_O(\theta+\pi)$.

The support function and the apparent diameter are linked by the following relationship, valid for every point O :

$$d(\theta) = h_O(\theta) + h_O(\theta+\pi) \quad \text{for} \quad 0 \leq \theta < \pi \tag{1}$$

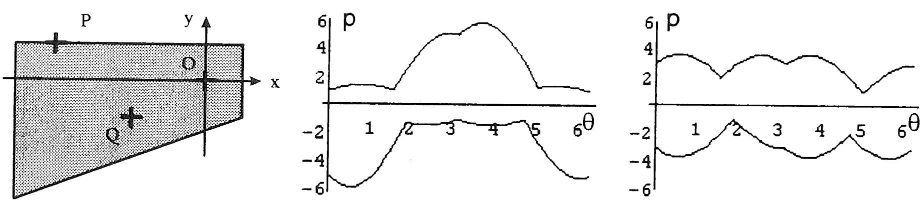
Let A be an interior point of the shape, with **polar** coordinates (p_{OA},θ_{OA}) where the index makes reference to origin O. The projection of A on the turning axis D_θ clearly yields a sine curve with equation :

$$p = p_{OA} \cos(\theta - \theta_{OA}) \tag{2}$$

If the origin's position is not ambiguous, this function will simply be denoted $p_A(\theta)$.

If a point Q, with polar coordinates (p_{OQ},θ_{OQ}), is taken as a new origin (see figure 2), it is easy to show that :

$$h_Q(\theta) = h_O(\theta) + p_{OQ} \cos(\theta - \theta_{OQ}) \tag{3}$$



A *B* *C*

Fig. 2A : The reference shape.

Fig. 2B : Graphs for $h_O(\theta)$ *and* $-h_O(\theta+\pi)$ *(origin O).*

Fig 2C : Graphs for $h_Q(\theta)$ *and* $-h_Q(\theta+\pi)$ *(origin Q). Fig 2C looks more « equilibrated ».*

A BARYCENTRICAL POINT OF VIEW

Let us consider three (non aligned) fixed points A, B, C in a shape K.

Any point M being given in K, vector $\vec{OM}$ can be decomposed, in a unique way, as

$$\vec{OM} = b\,\vec{OB} + c\,\vec{OC}$$

or $M = a\,O + b\,B + c\,C$ with quantity a defined by relationship $a + b + c = 1$.

Coefficients a, b, c are called the barycentrical coordinates of point M.

A very straightforward argument shows that

$$p_M(\theta) = a\, p_O(\theta) + b\, p_B(\theta) + c\, p_C(\theta) \tag{4}$$

If, for example, O is taken as the origin, its associated sine curve is « flat » (see figure 3).

In this case, formula (4) becomes $p_M(\theta) = b\, p_B(\theta) + c\, p_C(\theta)$.

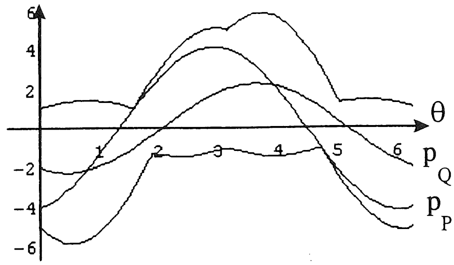


Fig. 3 : The sine curve of any point M of the shape can be obtained as a « weighted sum » of the sine curves of three fixed points. Here, these points are O, P(-4,1), Q(-2,-1), introduced on Fig. 2 (note that the sine curve associated to O is the θ axis).

HOUGH TRANSFORM

« Classical » Hough Transform for points

An origin O being chosen, Hough Transform (H.T.) associates, to any point A with polar coordinates (p_{OA}, θ_{OA}), the sine curve $\Gamma_{\theta_{OA} p_{OA}}$ with equation $p = p_{OA} \cos(\theta - \theta_{OA})$ in the (θ, p) coordinate plane called the **Hough plane**.

Remarks : This definition should be interpreted in connection with formula (2).

Every origin choice gives a specific H.T.

The original interest for H.T. is that, to a set of **aligned** points, is associated a family of sine curves which passes through a common point with coordinates (p_1, θ_1) in the Hough plane.

Moreover, the equation of the line on which the set of points is located (see figure 4).is

$$x \cos \theta_1 + y \sin \theta_1 = p_1$$

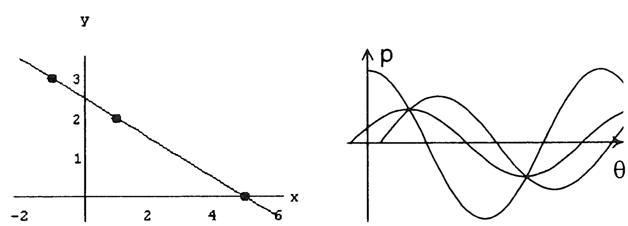


Fig. 4 : The H.T. of aligned points is a set of intersecting sine curves in the Hough plane.

Hough Transform for shapes

The H.T. of a planar convex shape is a **subset** of the Hough plane introduced above, which can be defined in three equivalent ways (see figure 3) :

(A) as the union of all $\Gamma_{\theta_o p_o}$ for points of the shape with polar coordinates (p_o, θ_o).

(B) as the region « swept » (i.e., generated) by all sine curves associated to a point which follows the shape's border.

(C) as the region situated between the graphs of functions $h_O(\theta)$ and $-h_O(\theta+\pi)$.

Remark : The boundary of the H.T. of a **polygonal** shape is made of portions of sine curves (see figure 2B). Indeed, the endpoints of the projected line segment on D_θ are always projections of vertices.

TWO APPLICATIONS

An efficient perimeter estimation

The perimeter of a **convex** shape K is given by the following formula (Santalo, 1976) :

$$P_K = \int_0^{2\pi} h_O(\theta)d\theta$$

which can be interpreted in the following way : P_K is a half of the area of the H.T. of K.

Its proof is easy, by integration on $[0, \pi)$ of both sides of relationship (1) and use of Crofton's theorem.

This connection gives a good way to obtain an accurate value of the perimeter of a convex shape by the computation of the area of its H.T. It is a reliable and efficient method compared to unpredictable direct evaluations based on contour tracking.

A new symmetry coefficient

Some domains of image processing, granulometry for example, need an efficient characterization of shapes circularity and/or symmetry, in order to get information for recognition or classification.

These shape properties are often expressed by coefficients defined on a certain scale, for example [0,1] ; a property is considered as fulfilled or not according to the values taken by the coefficients. For example, if A is the area and P is the perimeter of a shape, the quantity $4\pi A/P^2$, which takes its values on the scale [0,1], is a (rather simple) coefficient of circularity.

We want to show how H.T. can be used as a «heuristic method » in this domain.

Let us introduce first the « middle function » with respect to an origin O

$$m_O(\theta) = \frac{h_O(\theta) - h_O(\theta + \pi)}{2} \quad \text{for} \quad 0 \leq \theta < \pi$$

Remark : $m_O(\theta) = 0$ for $\theta \in [0, 2\pi)$ if and only if the shape possesses O as a symmetry center.

Point O can be considered as « well centered » if function $m_O(\theta)$ has « moderate » variations with respect to θ axis ; but these variations should be taken in a relative manner : if, for a certain θ, $d(\theta)$ is large, value $m_O(\theta)$ will tend to be relatively large too. This is why $C_O = \sup_\theta \frac{2m_O(\theta)}{d(\theta)}$ is a better centering measure than $\sup_\theta m_O(\theta)$, with $\frac{2}{d(\theta)}$ as weighting function. Now, the « best centered origin » O is a point which gives the smallest C_O , whence a new symmetry coefficient :

$$\varphi = \inf_O \sup_\theta \frac{2m_O(\theta)}{d(\theta)}$$

Remark: $0 \leq \varphi \leq 1/3$ with $\varphi = 0$ (resp. $\varphi = 1/3$) if and only one has a centrally symmetrical shape (resp. a triangle). The point for which φ is reached is the classical Minkowski point (Labouré et al, 1993), which is the best centered point in the shape.

Coefficient φ can be related to classical Minkowski coefficient ψ defined by :

$$\psi = \sup_O \inf_\theta \frac{h_O(\theta)}{h_O(\theta + \pi)} \qquad \text{in the following way :} \qquad \psi = \frac{1 - \varphi}{1 + \varphi}$$

RADON TRANSFORM

A natural extension of H.T. to gray level shapes is Radon Transform (R.T.), a basic tool in Computed Tomography (Toft, 1995), illustrated on figure 5.

A discrete version of R.T. can be defined as follows. Let K be a gray level discrete shape on a grid. To each point of K with gray level n, we associate, in a discrete Hough plane, a discrete sine curve with a gray level εn (where ε is a small constant), being understood that a summation of these gray levels takes place : the resulting « shape » is the (discrete) R.T. of K, often called its « **sinogram** » (see figure 5).

For the continuous R.T., replace summation by integration.

R.T. is linear and invertible : it is possible to characterize a shape by its sinogram even if it is non convex and defined in a « fuzzy » manner (in the biological domain for example).

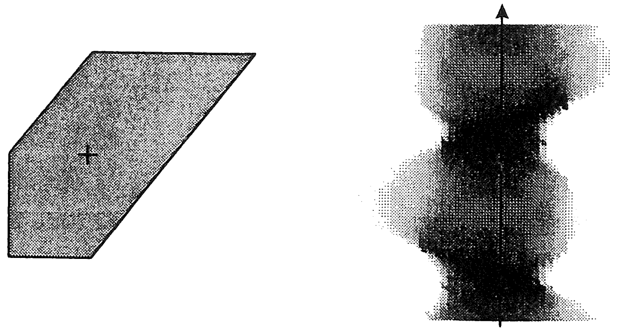


Fig. 5 : A polygonal shape and its sinogram. Some portions of sine curves are visible

CONCLUSION AND FUTURE ORIENTATIONS

We have shown, with two very different examples (perimeter computation, determination of a shape coefficient), that H.T. provides a new approach to shape study ; other application fields could be given, for example in connection to radial density (Fillère, 1995). Its natural enlargement, R.T., overcomes some limitations of H.T. ; its rich theoretical and practical framework can bring new answers (and new questions) in the domain of shape study.

We are working on another extension of H.T., where sine curves are replaced by other curves : significant results have already been obtained (Fillère et al., 1998 ; Becker, 1998).

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