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ON THE SUPERPOSITION OF RANDOM MOSAICS

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ABSTRACT

We compute the mean values of the area $a_{12\ldots m}$, the perimeter $h_{12\ldots m}$, the number of arcs $w_{12\ldots m}$ and the number of vertices $v_{12\ldots m}$ of a typical polygon of the superposition of m independent random mosaics. Some particular cases are considered. For definitions and basic formulas see Cowan (1980) and Santaló (1984).

SUPERPOSITION OF RANDOM MOSAICS

Cowan (1980) defines as characteristics of a random mosaic M, the mean values of the area a, the perimeter h, the number of arcs w and the number of vertices v of a "typical polygon" of M (suitably defined).

In Santaló (1984) we computed the characteristics of the random mosaic obtained by homogeneous random superposition of two independent random mosaics M_i of characteristics a_i, h_i, w_i, v_i $(i = 1, 2)$. The result was

$$\begin{aligned} a_{12} &= \frac{2\pi a_1 a_2}{2\pi(a_1 + a_2) + h_1 h_2}, \qquad h_{12} = \frac{2\pi(a_1 h_2 + a_2 h_1)}{2\pi(a_1 + a_2) + h_1 h_2} \\ w_{12} &= \frac{2\pi(w_1 a_2 + w_2 a_1) + 4h_1 h_2}{2\pi(a_1 + a_2) + h_1 h_2} \\ v_{12} &= \frac{2\pi(v_1 a_2 + v_2 a_1) + 4h_1 h_2}{2\pi(a_1 + a_2) + h_1 h_2}. \end{aligned} \tag{1}$$

If we superpose m independent random mosaics M_i $(i = 1, 2, \ldots, m)$ of characteristics a_i, h_i, w_i, v_i (always assuming that the superposition is random homogeneous), we get the following result:

Theorem 1. *The characteristics of the random mosaic obtained by superposition of*

m independent random mosaics M_i, are the following:

$$
\begin{aligned}
a_{1\ldots m} &= \Delta^{-1} 2\pi a_1 \ldots a_m \\
h_{1\ldots m} &= \Delta^{-1} 2\pi \{h_1 \mid a_2 \ldots a_m\} \\
w_{1\ldots m} &= \Delta^{-1} \big(2\pi \{w_1 \mid a_2 \ldots a_m\} + 4\{h_1 h_2 \mid a_3 \ldots a_m\}\big) \\
v_{1\ldots m} &= \Delta^{-1} \big(2\pi \{v_1 \mid a_2 \ldots a_m\} + 4\{h_1 h_2 \mid a_3 \ldots a_m\}\big)
\end{aligned}
\tag{2}
$$

where

$$\Delta = 2\pi\{a_1 \ldots a_{m-1}\} + \{h_1 h_2 \mid a_3 \ldots a_m\}$$

and $\{\ \}$ indicates "symmetric functions", i.e.

$$
\begin{aligned}
\{h_1 \mid a_2 \ldots a_m\} &= h_1 a_2 \ldots a_m + a_1 h_2 \ldots a_m + \cdots + a_1 \ldots a_{m-1} h_m \\
\{h_1 h_2 \mid a_3 \ldots a_m\} &= h_1 h_2 a_3 \ldots a_m + h_1 a_2 h_3 \ldots a_m + \cdots + a_1 \ldots h_{m-1} h_m \\
\{a_1 \ldots a_{m-1}\} &= a_1 a_2 \ldots a_{m-1} + a_1 \ldots a_{m-2} a_m + \cdots + a_2 a_3 \ldots a_m \,.
\end{aligned}
$$

Proof. By induction. For $m = 2$ the formulas (2) hold, since they coincide with (1). Assuming that they hold for m mosaics M_i applying (1) to the pair of mosaics $M_1 \cup M_2 \cup \cdots \cup M_m$ and M_{m+1}, a straightforward computation verifies that (2) holds for $m + 1$ mosaics.

CASE OF MOSAICS WITH THE SAME CHARACTERISTICS

If the random mosaics have the same characteristics a, h, w, v the formulas (2) take the form

$$
\begin{aligned}
a_m &= 4(m\Delta)^{-1}\pi a^2\,, \quad h_m = 4\Delta^{-1}\pi a h \\
w_m &= 4\Delta^{-1}\big(\pi a w + (m-1)h^2\big) \\
v_m &= 4\Delta^{-1}\big(\pi a v + (m-1)h^2\big)\,,
\end{aligned}
\tag{3}
$$

where

$$\Delta = 4\pi a + (m-1)h^2\,.$$

Consequences. 1. If $v = 4$, we have $v_m = 4$ for any m.
2. For $m \to \infty$ we always have $v_m \to 4$.

Examples. 1. For Poisson random mosaics (Miles, 1970; Santaló, 1976, p.57) we have $a = 4/\pi\lambda^2$, $h = 4/\lambda$, $w = v = 4$ and (3) gives a_m, h_m, w_m, v_m. For instance, we have $w_m = v_m = 4$ for any m.

2. For random mosaics of Voronoi type of the same characteristics, we have (Miles, 1970; Santaló, 1976, p.57), $a = 1/\lambda$, $h = 4/\lambda^{1/2}$, $w = v = 6$ and we have

$$w_m = v_m = 6 - \frac{8(m-1)}{\pi + 4(m-1)}$$

which is a decreasing function of m, from 6 to 4.

3. For random mosaics of Delaunay type, see Miles (1970), we have $a = 1/2\lambda$, $h = 32/9\pi\sqrt{\lambda}$, $w = v = 3$ and we get

$$w_m = v_m = 3 + \frac{32^2(m-1)}{32^2(m-1)+162\pi^3}\ .$$

4. Consider the regular mosaics of equilateral triangles ($w = v = 3$), squares ($w = v = 4$), regular hexagons ($w = v = 6$), or any affine transforms of them. By uniform random superposition of m such mosaics we get, respectively (according to (3)),

$$\begin{aligned} w_m &= v_m \text{ (triangles)} = 4 - \frac{4\pi a}{4\pi a + (m-1)h^2} \\ w_m &= v_m \text{ (parallelograms)} = 4 \\ w_m &= v_m \text{ (hexagons)} = 4 + \frac{8\pi a}{4\pi a + (m-1)h^2}\ , \end{aligned}$$

i.e. the mean number of vertices (equal to the mean number of sides) of a typical polygon is less than 4 for the superposition of triangular mosaics, equal to 4 for the superposition of mosaics of parallelograms and greater than 4 for the superposition of mosaics of hexagons. This gives a criterion for recognising if a given random mosaic is the result of superposition of mosaics of triangles, parallelograms or hexagons. Of course, the condition of w or v being less, equal or greater than 4 is only a necessary condition, not sufficient.

MOSAICS OF RECTANGLES

Formulas (1) apply to the mosaics obtained by random superposition of m non random mosaics (tessellations, i.e. arrangements of congruent polygons fitting together so as to cover the whole plane without overlapping). Then a_i, h_i, w_i, v_i are the area, the perimeter, the number of sides and the number of vertices of a polygon of the mosaic. The mosaics can be assumed moving in the plane without deformation with the kinematic density of integral geometry (Santaló, 1976).

Consider, for instance, the case of m mosaics M_i of congruent rectangles of sides δ_i, λ_i ($i = 1, 2, \ldots, m$) (formed by lines parallel to the x-axis at distance δ_i apart and the lattice of orthogonal parallel lines at distance λ_i apart). Then we have

$$a_i = \delta_i\lambda_i\ ; \quad h_i = 2(\delta_i + \lambda_i)\ , \quad w_i = v_i = 4$$

and for the mosaic obtained by random superposition of them we get

$$\begin{aligned} a_{12\ldots m} \text{ (rectangles)} &= \pi\Delta^{-1}\delta_1\ldots\delta_m\lambda_1\ldots\lambda_m \\ h_{12\ldots m} \text{ (rectangles)} &= 2\pi\Delta^{-1}\{(\delta_1+\lambda_1)\delta_2\lambda_2\ldots\delta_m\lambda_m\} \\ w_{12\ldots m} &= v_{12\ldots m} \text{ (rectangles)} = 4\ , \end{aligned} \tag{4}$$

where

$$\Delta\ (\text{rectangles}) = \pi\{\delta_1\lambda_1 \dots \delta_{m-1}\lambda_{m-1}\}\ 2\{(\delta_1+\lambda_1)(\delta_2+\lambda_2)\delta_3\lambda_3 \dots \delta_m\lambda_m\}\ .$$

For congruent mosaics of rectangles of sides $\delta_i = \delta$, $\lambda_i = \lambda$ we have

$$\begin{aligned}
a_{1\dots m}\ (\text{congruent rectangles}) &= \frac{\pi\delta^2\lambda^2}{\pi m\delta\lambda + m(m-1)(\delta+\lambda)^2} \\
h_{1\dots m}\ (\text{congruent rectanges}) &= \frac{2\pi(\delta+\lambda)\delta\lambda}{\pi\delta\lambda + (m-1)(\delta+\lambda)^2} \\
w_{1\dots m} = v_{1\dots m}\ (\text{congruent rectangles}) &= 4\ .
\end{aligned}$$

For mosaics of squares of side δ we have $\lambda = \delta$ and thus

$$\begin{aligned}
a_{1\dots m}\ (\text{squares}) &= \frac{\pi\delta^2}{\pi m + 4m(m-1)} \\
h_{1\dots m}\ (\text{squares}) &= \frac{4\pi\delta}{\pi + 4(m-1)} \\
w_{1\dots m} = v_{1\dots m}\ (\text{squares}) &= 4\ .
\end{aligned}$$

If $\lambda_1, \lambda_2, \dots, \lambda_m \to \infty$ the mosaics of rectangles tend to lattices of parallel lines at distances $\delta_1, \delta_2, \dots, \delta_m$ apart. Then, from (4) we deduce the following.

Theorem 2. *If m lattices of parallel lines at distances $\delta_1, \delta_2, \dots, \delta_m$ apart are superposed independently at random, the resulting random mosaic has the following characteristics*

$$\begin{aligned}
a_{1\dots m}\ (\textit{parallel lines}) &= \pi\Delta^{-1}\delta_1 \dots \delta_m \\
h_{1\dots m}\ (\textit{parallel lines}) &= 2\pi\Delta^{-1}\{\delta_2 \dots \delta_m\} \\
w_{1\dots m} = v_{1\dots m}\ (\textit{parallel lines}) &= 4\ ,
\end{aligned}$$

where

$$\Delta = 2\{\delta_1\ \delta_2 \dots \delta_m\}\ .$$

If the parallel lines are the same distance apart for all lattices, we have $\delta_1 = \delta_2 = \dots = \delta_m = \delta$ and so

$$\begin{aligned}
a_{1\dots m}\ (\text{equidistant parallel lines}) &= \frac{\pi\delta^2}{m(m-1)} \\
h_{1\dots m}\ (\text{equidistant parallel lines}) &= \frac{2\pi\delta}{m-1} \\
w_{1\dots m} = v_{1\dots m}\ (\text{equidistant parallel lines}) &= 4\ .
\end{aligned}$$

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