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IMPORTANCE OF THE CONNECTIVITY NUMBERS IN QUANTITATIVE IMAGE ANALYSIS

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ABSTRACT

Microstructure is often described quantitatively in two dimensional space from measurements on sections. But the morphology of materials must be defined in three dimensional space. In this paper we present the different parameters allowing to describe quantitatively a structure in three dimensional space without any hypothesis.

Key words

Three dimensional space analysis, connectivity number, Euler-Poincaré constants.

INTRODUCTION

The properties of a material depend mainly on its texture, i.e. its macrostructure or its microstructure according to the scale at which the constitutive elements are observed. Moreover the physical properties of a material concern the whole volume and thus they depend on the tridimensional characteristics of its texture.

To analyze the morphology of a material, it is necessary to know "a priori" :
- the number of present phases,
- the interfaces between the phases.

A texture can be intuitively described by parameters such as :
- quantity,
- size,
- number,
- shape.

Otherwise the spatial distribution of the elements of the structure is also accessible by describing :
- the geographical dispersion,
- the dispersion of orientation (anisotropy),
- the dispersion of size (size distribution measurements).

An intersection of the microstructure by a random plane allows to determine some of these characteristics. Some tridimensional parameters are accessible by an analysis in two dimensional space or in spaces of a lower dimension. But these parameters describe only the quantity and the size. To obtain a complete description of a microstructure, i.e. to accede also to the number and to the shape without restrictive hypothesis, it is necessary to realize an analysis of the microstructure in the three dimensional space.

The aim of this paper is to present the parameters allowing to describe quantitatively, without any hypothesis, a three dimensional structure. We shall begin by presenting the restraints on the analysis itself.

RESTRICTIVE CONDITIONS OF ANALYSIS

A structure cannot be described using any parameter in any conditions.

First, the local continuity of the structure is essential to insure the meaning of the measure. That is to say that the parameters must be independent of the scale of measurements or that the fineness of the analysis must be sufficient to have a large structure regarding the scale of analysis. In other words, the structure must not present a fractal aspect.

Secondly the whole frame of analysis (eventually subdivided in several frames) must be large regarding the scale of the structure. This means that the sampling must be sufficiently important to obtain a measure tending to a limit when the number of fields of analysis increase (ergodicity). This second condition is essential only for the local analysis of a microstructure, i.e.when this one is observed locally through a frame of measurements and not globally in all its entireness. This is practically always the case in materials science, in geology and in biology for example.

At last one can imagine a great number of parameters to describe a structure but they must possess a correct physical and mathematical meaning. For the stereological context, H. Hadwiger (1957) has proposed the conditions allowing to select these parameters. They are :

- Invariance by translation or rotation
  The morphological information obtained on a structure must be independent of the position of the frame of measurements. It is to be noted that the anisotropy can be characterized by parameters which, precisely, do not follow this condition of invariance by rotation.

- Homogeneity
  If the measurement is done at several magnifications on the same set, the results must be the same.

- Continuity
  A small deformation of the structure must not lead to large changes in the parameter measured.

- Additivity
  The condition of additivity is essential to calculate the means.
  By using set relationship we have :
  $$W(X) + W(Y) = W(X \cup Y) + W(X \cap Y)$$
  where $W(X)$ is the measure of the parameter W on the set X.

H. Hadwiger has shown that there exist four parameters corresponding to the previous criteria when the structure is defined in the three dimensional space. They are :

- the volume,
- the surface,
- the integral of mean curvature,
- the integral of total curvature.

The three first parameters are related to the quantity and to the size: these are metric properties. The last one depends on the number and on the shape of the surfaces : this is a topological property of the three dimensional space.

We shall present now the parameters which can be measured in this restrictive case and which allow a global description of microstructures.

METRIC PROPERTIES

If the hypothesis of stationarity is made when a structure is analyzed in local conditions, this allows to replace the measurements of a parameter by its probability. The probabilistic relationships of the stereology allow then to estimate :
- the volumic fraction, V_V,
- the specific surface area per unit volume, S_V,
- the integral of mean curvature, M_V.

As the material is not known in its integrality, it is sampled and the properties are brought back per unit volume in order to normalize them.

The use of these parameters is very important in materials science and in geology. For example one can cite the linear evolution of two physical properties : hardness and coercitive field, as a function of microstructural parameters specific surface area and mean free path (Rhines, 1985 ; Exner & Fischmeister, 1966).

TOPOLOGICAL PROPERTIES

The topological properties of objects are related to their boundaries. In R^3 space these boundaries are surfaces which can be characterized by their :
- number,
- orientation,
- edge,
- genus.

The surfaces present in a material are orientable and without edge. On the other hand their number and their genus can vary to the infinity.

The genus of a surface is the maximum number of closed curves that can be drawn on this surface maintaining its connexity (one says that a set is connex if, for all pair of points belonging to this set, one can draw at least one path totally included in the set). In other words, in R^3 space the genus of a surface corresponds to the number of sections that can be made through the set defined by this surface without cutting it in several parts (Fig. 1).

Among the topological parameters only the number of connectivity in the different spaces fulfills the Hadwiger conditions. The number of connectivity in the R^3 space, N_3, is a combination of the genus, g, and of the number, s, of the distinct surfaces present in the structure. One have :

$$N_3 = s - \sum_s g$$

We have, there, three topological parameters linked together :
- the connectivity number, N_3,
- the number of distinct surfaces, s,
- the genus of each surface, g.

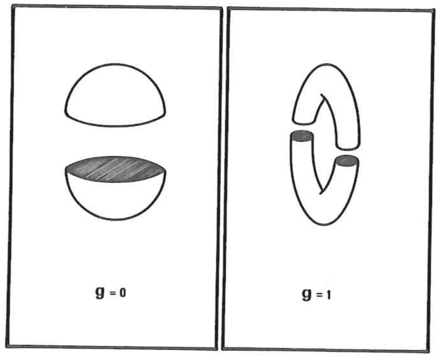

Fig. 1 : Genus of two surfaces : sphere and torus.

The connectivity numbers can be defined in the four spaces, R^0, R^1, R^2 and R^3 with a geometrical construction due to H. Poincaré (1912). Without entering in the details of this construction (Coster & Chermant, 1985), we shall give some brief indications on the way how to obtain the different connectivity numbers. In all cases the connectivity number of the set X in the space R^n is related by recurrence to the connectivity number of the sub-sets of X in the space R^{n-1}.

- The connectivity number, N_0, in the space R^0, of a set X is defined as the number of points of X belonging to the intersection of X by a grid of points.
- The connectivity number, N_1, in the space R^1, corresponds to the number of segments cut in X by its intersection with a straight line.
- The connectivity number, N_2, in the space R^2, is obtained by the successive intersections of X with a straight line scanning all the plane. It corresponds to the number of connex particles of X less the number of enclaves contained in it (Serra, 1982).
- The connectivity number, N_3, in the space R^3, is defined by the successive intersections of X by a plane scanning all the space. This last connectivity number can be only calculated in undertaking a three dimensional analysis of the microstructure.

One can note that, for bounded sets, there exists a simple relationship between the connectivity number, N, of a set and the connectivity number, N^c, of the complementary set :

$$\left| N^c + (-1)^j N \right| = 1$$

with j the dimension of the space where the set X and the complementary set X^c are defined.

Figure 2 presents three examples of objects in the space R^3 with their number of surfaces, their connectivity numbers and their genus. It can be seen on this figure that the physical number of objects is different from the number of surfaces, from the genus and from the number of connectivity.

Moreover the genus does not appear in the microstructural parameters defined with the Hadwiger criteria. The genus (as the number of objects or the number of surfaces) does not follow the condition of additivity. Then, the genus and the number of objects cannot be exactly obtained locally. This is shown on figure 3 where the relationship of additivity has been tested for two kinds of sets homeomorphic to the sphere. It can be seen that the only topological parameter which verifies this relationship with-

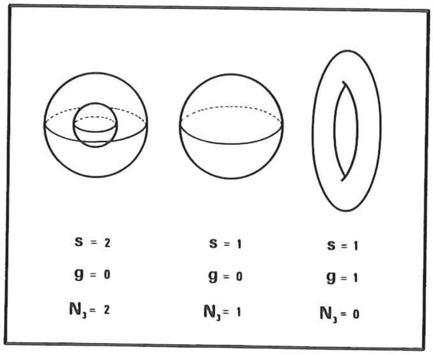

Fig. 2 : Number of surfaces, s, genus, g, and connectivity number, N_3, of a hollow sphere, a sphere and a torus.

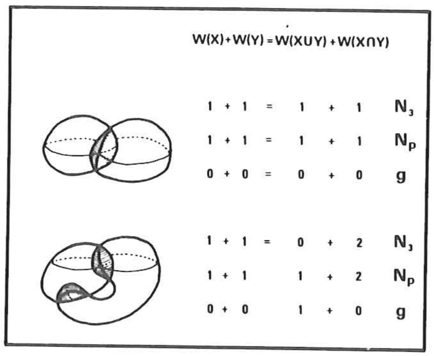

Fig. 3 : Testing of the relation of additivity for three parameters : the connectivity number, N_3, the physical number of particles, N_s, and the genus, g.

out any hypothesis is the connectivity number. It verifies also the relationships of additivity for the sets homeomorphic to the torus or to a hollow sphere or for any arrangement of these ones.

The connectivity number in the R^3 space is related to the integral of total curvature of the structure (or Gaussian curvature) by the relationship (Serra, 1982) :

$$G = \iint_S \frac{1}{R_1 R_2} dS = 4 \pi N_3$$

R_1 and R_2 being the main curvature radii in each point of the surface S.

ACQUISITION OF THE MICROSTRUCTURAL PARAMETERS

The automatic analysis carried out in sampling the structure to be analyzed by a set of points is particularly interesting. First the analysis is rapid and automatic ! Then the Euclidean space, which is very rich, is replaced by a space poorer but much more easy to manipulate : the number of neighbours of a point is finite, which is not the case for the Euclidean space. We have seen that all the microstructural parameters - V_V, S_V, M_V, N_V - can be obtained by counting : then we have only to investigate neighbourhoods (except for N_0).

The analysis of a digitalized structure can be made in using two methods :
- adaptation of the Euler-Poincaré method to a discretized space,
- use of the Euler relation.

In order to simplify we will present there only the Euler method. Information on the Euler-Poincaré method can be obtained in Coster & Chermant (1985) and in Serra (1982). Firstly established to describe the surfaces of polyedrons, the Euler relation is generalizable to a discretized space in which configurations are tested. The connectivity number of a set defined in a space j is given by :

$$N_j = \sum_{i=0}^{j} (-1)^i \, n_i$$

with n_i the number of elements of dimension i.

For example in the space R^3 we have :

$$N_3 = n_0 - n_1 + n_2 - n_3$$

with n_0 : number of vertices,
n_1 : number of edges,
n_2 : number of faces,
n_3 : number of elementary volumes.

The shape of the elements depends obviously of the type of grid of analysis. For clarity's sake we shall use for the examples a square (or cubic) grid, considering that each point possesses 4 neighbours in R^2 space (the diagonal points being excluded) and 6 neighbours in R^3 space. The Euler relation is illustrated in a digitalized space of analysis on figures 4 and 5. For R^3 space, the three examples correspond to objects homeomorphic to the sphere, to the torus and to a hollow sphere. A combination of these three objects allows to construct any tridimensional structure. The connectivity numbers obtained in the different spaces of analysis are then directly related by simple relationships (J. Serra, 1982) to the four quantitative parameters in local analysis :
- volumic fraction, $V_V \longleftrightarrow N_0$
- specific surface area, $S_V \longleftrightarrow N_1$
- integral of mean curvature, $M_V \longleftrightarrow N_2$
- integral of total curvature, $G_V \longleftrightarrow N_3$

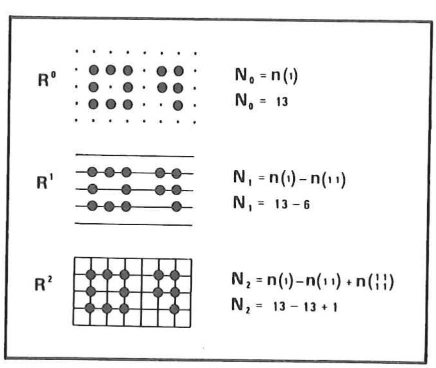

Fig. 4 : Illustration of the extension of the Euler relation in R^0, R^1 and R^2

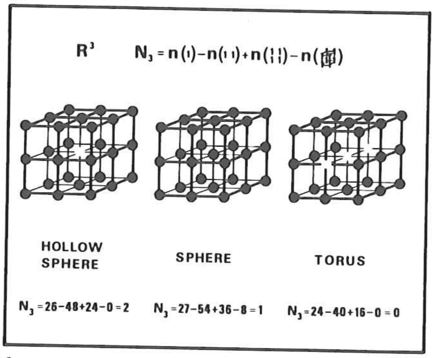

Fig. 5 : Illustration of the extension of the Euler relation in R^3.

MODELIZATION OF THE STRUCTURE

These four parameters are the only ones which allow to describe exactly a structure without any hypothesis or without any previous information on the structure. On the other hand if we possess information on the structure before the quantitative analysis, the qualitative observations that we have allow a modelization. Then an interaction can arise between the observation of the structure and its analysis. This interaction gives quantitative morphological information unaccessible by another way.

For example, let us take a material with two phases in which the imbricated degree of the phases can present two limit cases and an infinity of intermediate configurations.

Totally dispersed structures

For a granular structure (Fig. 6) the simplest model is based on particles homeomorphic to the sphere. In this case, for the surfaces of separation with the complementary set, the value of the genus is zero. To each particle corresponds one surface and, in this case, the connectivity number is nothing else that the physical number of particles, N_P :

$$N_3 = N_P$$

Then the number of connectivity allows to accede to the physical

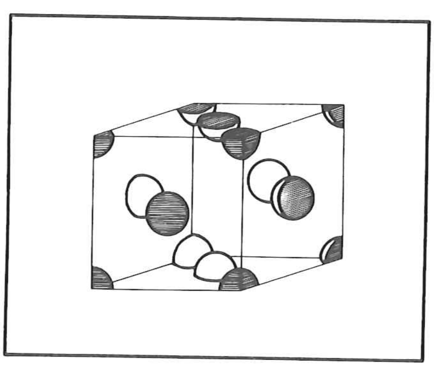

Fig. 6 : Totally dispersed structure.

number of particles without any other hypothesis than the homeomorphism to the sphere (the convexity of the particles being not necessary). We have to note that the "classical" stereological methods used to determine the number of particles per unit volume (even those which use serial sectionning) are all based on the hypothesis that the structure is composed of convex disconnected sets. The number of objects is only locally accessible in this case, as it is the only one where the condition of additivity is verified. This is also true in the limit case where the volume fraction of one of the two phases is zero (polycrystalline material).

Totally interconnected structures

In the case of a totally interconnected structure (Fig. 7), there exists only one surface of separation between the two phases. The number of connectivity is then related only to the genus of this surface :

$$N_3 = 1-g$$

One notes that, for this case, the size in the R^3 space, is not accessible from the connectivity number unlike the previous case ($\bar{v} = V_V/N_V$). Anyway, the notion of size defined in R^1 (mean free path) keeps always its meaning.

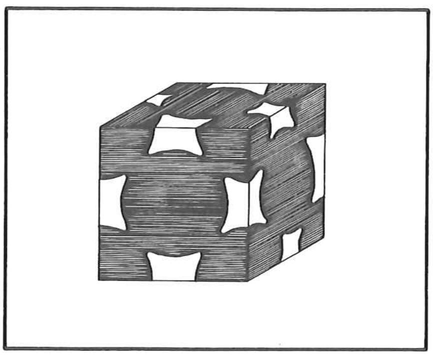

Fig. 7 : Totally interconnected structure.

Intermediate structures

A granular structure during sintering is a good example for intermediate structures. The porous space, initially interconnected (Fig. 7), disappears progressively by dispersion (Fig. 6). To understand the change in the morphology of such a material, it is necessary to follow the solid/ pore interface.

The connectivity number is related to the integral of total curvature: it allows then to appreciate the relative ratio of the saddle surfaces with respect to the concave or convex surfaces when branching of the interfaces arises by a complex way.

CONCLUSION

To analyze a three dimensional structure without any previous hypothesis on its shape, the connectivity number measured in the different spaces R^0, R^1, R^2 and R^3 is the only parameter which takes into account the characteristics of this structure by a mathematically exact way. One can say that:

- N_0 (or V_V) informs on the quantity,
- N_1 (or S_V) defines a kind of size,
- N_2 (or M_V) is linked to the irregularity of the structure and to its number of concave surfaces,
- N_3 (or G_V) is linked to the number of branchings of the structure.

To complete these informations, an interaction between the analysis and the observation of the structure can allow to modelize the structure unambiguously.

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