ACTA STEREOL 1987; 6/SUPPL II: 69—78
ISS COMMEMORATIVE-MEMORIAL VOLUME
DISCOURSE

"Looking into space:
an introduction to stereology"

Hans Elias

"My greatest discovery in mathematics
was Stefan Banach"

Hugo Steinhaus

HUGO STEINHAUS - AN UNKNOWN STEREOLOGIST ?

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ABSTRACT

The paper is devoted to the outstanding figure of Hugo Steinhaus, co-originator of the Polish School of Mathematics and to a short review of the results he had achieved in the field which has now become an independent scientific discipline: stereology.

Keywords: Empirical curve, length of an arc, fractals, stereology.

INTRODUCTION

First some information concerning Steinhaus "curriculum vitae" selected from his "Autobiography" /1973a/, published after his death, and from the monograph written by Kuratowski /1973/. Hugo Dyonizy Steinhaus /1887-1972/ was born in Jasło, a small town in the South of Poland. There he finished the secondary school of the grammar school type. In 1905 he started his studies at the Jan Kazimierz University in Lwów, which he continued in Göttingen, in Germany, where in 1911 he obtained the doctor's degree after submitting a dissertation on "New applications of Dirichlet´s principle". His supervisor was D. Hilbert. In 1920 he was appointed professor at the University in Lwów. Together with Stefan Banach they founded the famous Lwów school of mathematics which attracted numerous, at that time young mathematicians, among them M.Kac /one of Steinhaus´ many students/ and S. Ulam /1969/. Steinhaus described his first encounter with S. Banach in the following words: "One summer evening when walking along the Planty /a green belt surrounding the Old City - Authors´note/ in Cracow I overheard a conversation or rather only a few words: the term Lebesgue´s integral" was so surprising that I came up to the bench and introduced myself to the participants of the discussion: they were Stefan Banach and Otton Nikodym who had been talking about mathematics". Kuratowski /1973/ writes: "this was how Banach «became discovered» . This encounter had almost immediate scientific consequences: Steinhaus informed Banach about a certain problem on which he had been working for quite a long time, and a few days later, to Steinhaus´surprise - Banach called presenting the ready solution. In that way Banach´s first publication appeared with Steinhaus as co--author /1918/".

Steinhaus´later years are described in his autobiography /1973a/: "At the outbreak of the Second World War I found myself together with my wife, daughter and son-in-law near the River Prut, at a place called Kamień Dobosza. We came back to Lwów, where soon the University started its activities: besides the staff members attached to the University until that moment, mathematicians, refugees from the territories occupied by the Germans / Saks, Knaster, Wojdysławski,Marczewski and others/ joined the University. I was appointed professor at the Chair of Higher Analysis and staff member of the Kiev Academy. The uncertainty of the situation paralysed any scientific activity./.../ Uncertainty changed into sorrow after Paris has been taken, and into a tragedy when the German troops crossed the frontier at the end of June 1941. On July 4th /on that day many eminent representatives of Polish science and culture were shot - Authors´note/ I left my flat at 14 Kadecka str. and until November 26th together with my wife stayed at my sister´s flat. /.../ Thanks to prof. Bulanda / former Rector of the Lwów University/ and Mrs. Morska-Knaster we found a safe lodging in Osiczyna near Lwów /.../. From November 16th, 1941 until August 26th, 1945 I existed under the name of Grzegorz Krochmalny, a peasant from the vivinity of Przemyśl, whose birth certificate was procured for me by Tadeusz Hollender, the poet. He himself was shot by the Germans at the Pawiak prison a few years later. /.../ On July 13th, 1942 we both moved to Berdechów, near Stróże, in the Gorlice district. There, together with my brother-in-law, an engineer, we continued secret teaching on a rather large scale - our pupils were mostly sons of peasants and railway men. While staying at Berdechów, in 1945, after the defeat of the Germans, I menaged to prepare two research studies: «On the problem of the electricity tariff» and «On halving a body by a plane»".

After the war Steinhaus moved to the University in Wrocław, where, for the second time in his life, he became the co-founder of a school of mathematics. He was appointed full member of the Polish Academy of Sciences and was granted the title of "doctor honoris causa " by some universities.

Steinhaus´interests and creative activity concentrated around the fields such as: theory of trigonometric series, theory of real functions, orthogonal series, calculus of probability, topology. The list of Steinhaus´scientific publications /1973b/ comprises 247 items, about half of which referring to the application of mathematics. Łukaszewicz /1973/ notices that a remarkable feature of Steinhaus´ research studies, especially in the field of application of mathematics, was solving "conflict situations" by means of a "pragmatic division" which was advantageous for both sides. Examples of such problems are: /a/ the line of ethnic balance, /b/ fixing of the price of mine timber, /c/ electricity tariff, /d/ acceptance of commodity goods, /e/ establishment of paternity.

Let us now concentrate on Steinhaus' publications referring to the field now called after Elias /1963/ stereology. Their number is a little more than ten, including some re-editions intended to the reader who might try to apply the proposed solutions in practice, in particular for geographers, mineralogists and physicians, thus, in general, non-mathematicians. Their subject matter is characteristic: /a/ length of empirical lines, /b/ surface morphology and /c/ location of foreign bodies in live organisms. What regards problem /a/ Steinhaus had actually only a few, but very prominent predecessors, such as Cauchy and Crofton. What regards problem /c/ Steinhaus was the forerunner of a method which is now called tomography.

GENERALIZATION OF CAUCHY´S THEOREM

Steinhaus /1973a/ recollects: "The problem of length measurement came to my mind when my daughter was told to measure on the map the length of the Vistula, what is a very instructive task for the parents". Steinhaus´ first two papers referring to length measurement were published in 1930. The starting point of his considerations /1930a/ was the quotation of Cauchy´s theorem /1832/: "The length of an arc of a plane curve is equal to $\pi/2$ times the mean length of its total projection in all directions":

$$L = \frac{\pi}{2} \cdot \frac{1}{\pi}\int_0^{\pi} W/\alpha/d\alpha = \frac{1}{2}\int_0^{\pi} W/\alpha/d\alpha \quad . \qquad /1/$$

Steinhaus proved this theorem using the trigonometric method and parametrization of the curve and applying the notion of Riemann´s integral. He notices that it can be proved for an arbitrary rectifiable curve by introducing the concept of Lebesgue´s integral. He remarks also that Cauchy´s theorem holds for spatial curves, but the projection of the curve must be made on all the planes and the mean value of the length of the total projection of the curve multipled by $4/\pi$.

ESTIMATION OF THE LENGTH OF A CURVE

Steinhaus solves the problem of the estimation of the length of a curve /1/ for the first time, applying the trigonometric method. He takes into consideration n directions disposed uniformly at $2\pi/n$ intervals. The following estimate is obtained:

$$\bar{L} = \frac{\pi}{2n} \sum W_i \qquad /2/$$

Detailed calculations are carried out for a segment of a straight line and for $n = 6$. He proves that the obtained estimation is valid for any rectifiable curve. Moran /1972/ quotes this estimation in the form:

$$\pi \cos \frac{\pi}{n} /2n \sin \frac{\pi}{2n}/^{-1} \leqslant \frac{\bar{L}}{L} \leqslant \pi/2n \sin \frac{\pi}{2n}/^{-1} \quad . \qquad /3/$$

For $n = 6$, 10, 20 the limits of estimation are equal to /0.9770, 1.0115/, /0.9918, 1.0041/ and /0.9979, 1.0010/, respectively. Moran states: "Unless the line L is nearly straight, the actual standard error of estimate will be substantially smaller than these limits suggest". Further on, but this time referring to the estimation of specific surface: "Thus the problem of estimating the specific interface area is not difficult even when the α phase has a general distribution". It is worth noticing that the starting point of Moran´s considerations /1944/ concerning the measurement of the surface areas of convex bodies was also Steinhaus´ publication /1930/. Moran discarded the trigonometric method in favour of a more convenient vectorial method. Kendall and Moran /1963/ in their monograph mentioned Steinhaus´ results discussed above.

STEINHAUS´ GRID

Steinhaus /1930/ gave a practical method of an approximate determination of the length of a line by means of a square grid of size d, drawn on a transparent sheet. He called it a "lengthmeter". He also published his method in a periodical for geography teachers /1931/ and in a journal for cartographers /1932/. Steinhaus´ grid was produced in 1930 by the publishing house Książnica Atlas as a teaching aid. Fig.1 comes from Stein-

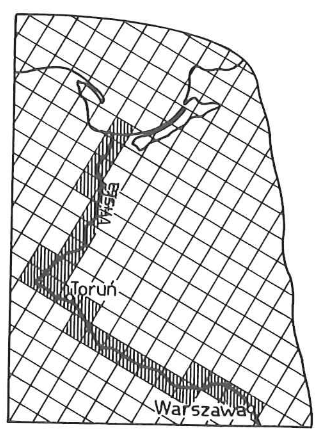

Fig. 1. Steinhaus´ grid.

haus´study /1931/. It comprises the contour of the Vistula from Warsaw down to its estuary into the Baltic Sea. Its "stereological" content is evident. It should be noticed, however, that Steinhaus´ grid was spaced d = 3.82 mm, so that the counting of the squares having parts in common with the river contour after threefold application of the grid gave as a result the length of the river contour directly in millimeters. Multiplying this number by the scale of the map enabled to obtain the length which was to be determined.

NEW DEFINITION OF LENGTH

Steinhaus /1930a,1930b/ gave a new definition of the concept of length: "Let us consider an arbitrary plane set A and determine Lebesgue´s measure $W/\alpha/$ of a projection of this set on a straight line / the segments covered many times should be taken into consideration as many times/ and accept

$$\frac{1}{2}\int_{0}^{\pi} W/\alpha/\, d\alpha \qquad /4/$$

as the length of the set under consideration". Steinhaus pointed to the following advantages of this definition: 1° It is independent of the parametric representation of the boundary of the set A, 2° It retains the invariance of length with respect to motion in the plane. 3° It does not require the introduction of kinematic concepts of the start and the end points. 4° It has a wider range of applicability. This last property of the above definition was discussed by Sherman /1942/. The definition of a surface area was given by Steinhaus in an analogous way. It is worth noticing that Steinhaus understood the mean length of the curve projection in the sense of the integral calculus. He did not make use of Crofton´s results /1868/ with which he was familiar. He would have obtained then equivalent formulation of the definition of length.

LENGTH OF THE ORDER m OF EMPIRICAL LINE

Finally, the length of empirical lines in Steinhaus´ approach/ 1932, 1949, 1954/. At that time the knowledge of that subject was considerably smaller. Steinhaus differentiates between curves given analytically and the empirical ones.Examples of the latter can be found in the first place in geography. He does not quote Penck´s monograph /1894/, in which the so--called generalization degree of the maps and the effect of the map scale on this degree were considered. On the other hand, he makes use of Goldschlag´s /1913/ investigations who criticized the then known methods of measuring the length of curves on maps and the effect of the subjective factor on the measurement results. Steinhaus introduced the concept of a length of the order m of the curve A, what enabled him to look "in a different way" at the effect which he called the "paradox of length". Steinhaus states that: "Length is a discontinuous functional. This means in plain words that we can trace in the vicinity of any rectifiable arc A another arc A´whose length exceeds an arbitrary previously prescribed limit or even is infinite. This fact is something more than a mathematical curiosity: it has practical consequences. When measuring the left bank of the Vistula on a school map of Poland, we get a length which is appreciably smaller than the one read on the map 1:200 000. The same difficulty arises when measuring such objects as contours of leaves or perimeters of plane sections of trees: the result depends appreciably on the precision of the instruments employed. A statement nearly adequate to reality would be to call most arcs encountered in nature not rectifiable. This statement is contrary to the belief that not rectifiable arcs are inventions of mathematicians and that natural arcs are rectifiable: it is the contrary that is true".

The starting point of Steinhaus´ mathematical considerations was the representation of Crofton´s formula /1868 / in the form:

$$|A| = \text{Length of } A = \frac{1}{2}\sum_{k=1}^{\infty} k\,|A_k| \qquad /5/$$

where $|A_k|$ is a measure in the sense of Lebesque on Crofton´s plane $/\vartheta, p/$ of a set of straight lines L intersecting A exactly at k points. After introducing the function a $/\vartheta, p/$ representing the number of the intersection points of the straight line L with the curve A, /5/ becomes transformed to:

$$|A| = \text{Length of } A = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{0}^{\pi} a/\vartheta, p/\ \, d\vartheta\, dp \qquad /6/$$

where the double integral is understood in the sense of Lebesque. Basing on /5/:

$$|A|_m = \text{length of order m of } A = \frac{1}{2}\sum_{k=1}^{m} k\,|A_k| \qquad /7/$$

or on the basis of the formulation of Crofton´s theorem in the form /6/:

$$|A|_m = \text{length of order m of } A = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{0}^{\pi} a^{/m/}\,/\vartheta, p/\ d\vartheta dp \qquad /8/$$

where $a^{/m/} = a$ for $a \leqslant m$ and $a^{/m/} = m$ for $a > m$. In case $m = 2, |A|_2$ represents the length of a convex envelope of the curve A. When A is a convex curve $|A|_2 = |A|$, $|A|_\infty$ is the classical length. The above result was quoted by Santaló /1976/. Fast and Götz /1952/ have proved that when $a/\vartheta, p/$ is bounded from above the integral /6/ cannot be replaced by Riemann´s integral either, unless the curve A is Jordan´s arc. According to Perkal´s /1958/ critical remarks the length of the order m can be determined for any empirical curve and it is easy to measure. The drawback of this concept is its too weak relationship with the classical length when $m > 2$. The length of the order m similarly as the classical length depends on the degree of generalization of the curve, that is, e.g. on the fact whether the curve is seen with naked eye or in a microscope. Last of all, the length of the order m is not an additive functional. Perkal introduces the concept of length of the order ε , where $\varepsilon > 0$.

Steinhaus /1949/ introduced the concept of relative length. "Let us write $|A|_m$ for the length of the order m of A; $|A|_m$ is always finite, but $\lim_{m \to \infty} |A|_m$ is infinite if A is not rectifiable. It may happen that the arcs A and B are both not rectifiable, but the limit

$$\lim_{m \to \infty} |A|_m \;/\; |B|_m = c \qquad /9/$$

is a finite number c. We may the write $|A| \;/\; |B| = c$, which means that A is c times as long as B. The device of a transparent sheet carrying parallel lines makes it possible to compute c with any desired accuracy in such practical cases as the comparison of the respective lengths of both banks of the Vistula."

It appears that so far no comparative analyses of the concept of length according to Steinhaus and of the concept of fractals have been carried out. The most complete und uniform presentation of Steinhaus´results can be found in his study /1954/. In the above discussed review the authors of this paper based on his early works.

MORPHOLOGY OF SURFACES

The problem of the morphology of surfaces and especially its simple and sensible quantitative estimation in practice had been one of the important and vital problems in geography long before the quantitative development of fractography and of the materials science which was to follow.

Steinhaus proposed to the geographers a quantitative estimation of the surface configuration of a territory Ω under consideration by means of determining two indices: that of the average steepness /1947a/ and the index of the vertical configuration /1947b/. When formulating his proposals Steinhaus restricted them to the utilization of information given on the maps and to information which can be obtained basing on the maps.

The index of average steepness is simply the "mean" tangent of the inclination angle of the terrain:

$$\mathrm{tg}\,\varphi = \frac{h \sum_i^n L_i}{|\Omega|} \qquad /10/$$

where h - difference in the height of the terrain between successive contour lines, $\sum L_i$ - sum of the contour lengths, $|\Omega|$ - area of the terrain Ω .

To introduce the index of the vertical configuration of the terrain Steinhaus takes into account its j-th vertical cross-section. The profile of the terrain $y_j = f_j/x/$ has its successive maxima and minima: y_{j0}.... y_{ji-1}, y_{ji}.....y_{jn}. The successive differences in the height of the profile are: $\Delta y_{ji} = y_{ji-1} - y_{ji}$. As an index of single vertical configuration of the j-th profile Steinhaus takes the expression

$$\mu_j = \sum_i^{n_j} \sqrt{|\Delta y_{ji}|} \quad j = 1\ldots\ldots m \qquad /11/$$

and as the index of the vertical configuration of the terrain

$$M = \frac{\bar{\mu}}{\sqrt{|\Omega|}} \qquad /12/$$

where $\bar{\mu} = \frac{1}{m} \sum_j \mu_j$.

LOCATION OF FOREIGN BODIES IN LIVE ORGANISMS

A few years after the discovery of the X-rays the naturalist and philosopher E. Mach put forward the concept of X-ray stereoscopy. Hasselwander, an anatomist from Erlangen, already during the First World War performed many operations by means of vision obtained by putting two appropriately taken X-ray photos into a stereoscope. Steinhaus was well acquainted with other methods, too, including the computational ones, the object of which was to locate foreign bodies in human organisms. In Steinhaus´ opinion the common drawback of these methods was that they did not provide an answer to the question where? since that question can be only answered with the word: "here". When we are looking at a stereoscopic image, and next at the ailing man, we are not able to answer the question: where? in a definite way, not requiring any further comments.

Steinhaus /1939, 1952/ supplied an unusually simple solution. It is based on a phenomenon which nearly everybody can watch every day. Steinhaus writes: "If we set up a glass panel in a vertical position on a horizontal board and drive two nails into that board, one behind the glass panel and the other one in front of it, we can try to arrange it so that the reflection of the first nail in the glass panel /which in general appears at the side of the other nail/ accurately meets the other, real nail behind the glass panel. If now we put a lump of plasticine on the nail behind the panel, then - looking at the glass panel - we shall see the lump of plasticine, to be more precisely, its surface, but inside it the nail will be visible as if the plasticine were transparent. This is why we may now reach with a knife behind the glass panel and easily cut off part of the plasticine so that the edge of the nail head be now only at a milimetre distance from the plasticine surface. Then, on a command, we may hit that edge with a knife. It is one of the most attractive games I know. It provides an answer to the question: where?".

Steinhaus´ observations led to the construction of an instrument which facilitated the operations in which foreign bodies were removed from human organisms. Such instruments were in use. The first operations were performed in King Jan III Sobieski Hospital in Lwów before the war. Two operations consisted in removing fragments of needles from the patient´s hands, and the third one - in removing a bullet situated in the vicinity of the pelvis.

The question arises, if Steinhaus´ idea has really become outdated so quickly?

FINAL REMARKS

It appears that Steinhaus´ ideas did not meet with proper response in the development of stereology which was to follow. The question can be raised again whether they really have become outdated so quickly? Steinhaus´

"Selected papers" /1985/ have been published recently. His monograph includes nearly 900 pages, but not a single publication in the field of stereology has been listed there.

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