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A GENERAL METHOD TO SOLVE THE INTEGRAL EQUATIONS OF STEREOLOGY

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ABSTRACT

The scope of stereology is to infer characteristics of features of a three-dimensional structure from measurements on two-dimensional (usually planar) sections. A feature is characterized by one or more variables, x, with a probability distribution function X(x). In a two-dimensional section one observes "prints" of the three-dimensional features intersected, which are characterized by variables, y, with a measurable distribution Y(y). Provided that the variables, x, completely define the features, it is possible to derive a relation between the two distributions, in the form of an integral equation giving explicity Y(y) as an integral containing X(x). The integral equation is usually unsolvable for X(x) by analytical means.

In the paper, a general method is developed to solve equations of this type. The basis of the method is the discretization of the two distributions in classes, or intervals, and the linearization of the relation between them, which can then be solved for X(x) by standard matrix methods. The linear relation between the two distributions involves coefficients which can be calculated with any degree of accuracy but depend on the chosen amplitudes of the intervals.

The stability of the solutions can easily be assessed through the method developed, and various parameters are introduced with which the stability can be evaluated.

The method is applied to the determination of the length distribution of oriented lines in a plane from the measurement of the number of points of intersection with test lines of various orientations.

KEY WORDS: integral equations, stereology, stability of solutions.

INTRODUCTION

The problems that arise in stereology (e.g. DeHoff and Rhines, 1968; Underwood, 1970) can be stated in the following way, where, for simplicity, we consider one-variable problems. A 3D structure contains (invisible) features of a given type, e.g. particles, boundaries (surfaces), lines (edges), dihedral angles, etc., each of which is characterized by the value of a variable x (in general, more than one variable may be considered). Examples of variables x are: particle dimensions or volume; surface area, curvature and orientation; line length, curvature and orientation; dihedral angle and orientation of its edge. The features (of a given type) have different values of x, and we assume that x is a continuous variable. It is then possible and convenient to define a distribution function, X(x), such that X(x)dx is the "quantity" of features, per unit volume of the sample, having values of the variable in the interval x, x+dx (shortly, interval x,dx). By quantity is meant, depending on the feature and on x, the number, volume, surface area or length of the features in the interval x,dx in a reference volume. Frequently, X(x) is defined as the fractional quantity per unit interval of x, relative to the total quantity of features: X(x)dx is then the probability of a feature having a value of the variable in the interval x,dx. In some problems the objective is to obtain the total quantity of features per unit volume, i.e., the features are not classified. These can be regarded as particular cases of the more general situation in which the particles are classified by the x variables.

The scope of stereology is the determination of X(x) from measurements on two-dimensional, usually planar, sections of the sample. These are termed "test planes". In some problems, the measurements are done on straight lines drawn on the planar sections. In such cases, we may say that the 3D sample is intersected by "test lines". Test lines can also be used simply to study features in a two-dimensional structure.

The test plane intersects some of the 3D features and produces a recognisable "print" of the features intersected. The print is characterized by a measurable variable y, which has a distribution Y(y), such that Y(y)dy is the "quantity" (or fraction relative to the total) of prints, per unit area of the test plane, which have the variable in the interval y,dy. In general, more than one variable y may be considered. Examples of y are: area of closed curve, length, curvature and orientation of lines; angle. The term "quantity" has the same meaning as before and can denote number, area or length of the prints (or the corresponding fraction).

In the case of "test lines", the prints are necessarily points or segments between successive points; y is defined as the distance between adjacent points (length of segments) and Y is the number (or fraction) of segments with a given length, per unit interval of length.

The test planes and test lines can be chosen with a fixed orientation and random location. The orientation is defined by n , the unit normal to the test plane or test line*. When the orientation is changed, a different distribution of "prints" may result, i.e., Y(y,**n**), in which case we say that the 3D features are oriented (otherwise, the structure is non-oriented and the features are randomly (uniformly) oriented). Frequently, in this case where different orientations **n** are used to test the sample, the prints are not classified, i.e., any print, whatever the value of y, is counted. Then it is the distribution Y(**n**) that is of interest, Y(**n**) being the quantity of prints for orientation **n**.

In other problems the orientation of the test planes or lines is random, in the sense that equal weight (equiprobability) is given to all orientations. If the structure is oriented, sections of various orientations have to be analysed, but if it is non-oriented any orientation of a section will be representative. In such cases, Y is defined as an average for all (equiprobable) orientations.

In general, both the 3D features and their prints may require more than one variable. In addition, as discussed above, the consideration of the effect of orientation of the test plane or line introduces additional variables in Y. The distribution functions are defined in a way similar to that used for one variable problems. For example, $X(x_1 , x_2)$ is defined such that $X(x_1 , x_2)dx_1dx_2$ is the

* No distinction is made between opposite directions, **n** and -**n**.

quantity of features with the variables in the intervals x_1, dx_1 and x_2,dx_2.

Stated now more precisely, the scope of stereology is the determination of the distribution X(x) from the measured distribution Y(y), where x and y indicate collectively the variables used in the characterization of the 3D features and of the 2D prints, respectively. Among y we may include the variables that define the orientation of the test planes (or test lines) used to produce the print.

It will be shown in the following section that an integral equation can be written relating the two distributions X and Y, provided that the variables x are adequately chosen. A general method of solution of the integral equation to obtain the distribution X(x) from an experimentally determined Y(y) is developed in the following section followed by an application to the determination of the orientation distribution of lines in the plane. Finally, an evaluation is made of the stability of the solutions obtained by the method developed. It will be shown that in general the stability may be poor but this is an intrinsic feature of the problem, and only precise experimental sampling and measurement can lead to more accurate results.

THE INTEGRAL EQUATIONS OF STEREOLOGY

In what we shall call well-defined problems of stereology, the variables x contain enough information on the features and an equation can be written relating the two distributions X(x) and Y(y). This happens when:

a) The quantity P(x)dx of features x,dx that are intersected can be written in the form

$$P(x) = X(x) \, . \, P_0(x,y)$$

meaning that P(x)dx is proportional to the quantity of features x,dx and to a function $P_0(x,y)$ which, in turn, is proportional to the relative probability of intersection of a feature x; the dependence of P_0 on y may occur in the case of the variables y related to the orientation of the test plane or test line, but otherwise P_0 is a function of the variables x exclusively.

b) The probability that an intersected feature x gives a print y,dy can be written in the form

$$P_1(x,y)dy$$

meaning that this probability is, for a given choice of the y variables, completely determined by the x variables.

Under these conditions, the distribution Y(y) is given by the following integral over all values of x

$$Y(y) = \int X(x) \, p(x,y)dx \tag{1}$$

where

$$p(x,y) = P_0(x,y) \, . \, P_1(x,y) \tag{2}$$

is a function that can be established by theory. Note that p(x,y) is non-negative for all values of x and y.

The best known examples of problems in stereology that give equations of type (1) are in the field of oriented distributions of surfaces and lines which are sectioned by test planes (or test lines) with a fixed orientation. The relevant equations were derived by Hilliard (1962). The variables that define the orientation of the test planes or lines are the y variables, while those defining the orientation of the three-dimensional features (surfaces, lines) are the x variables. In such cases P_1 is a constant and $p(x,y)dx$ is the probability that a feature x,dx is intersected by a test plane with orientation y.

Another example is provided by dihedral angles, x, with distribution X(x), and its relation to the distribution Y(y) of planar angles y in sections. In this case $p(x,y)dy$ is the probability that a dihedral angle, x, originates an angle in the interval y,dy. This problem was treated in detail recently (Fortes and Ferro,1988), using the general method that will be developed below.

An important class of problems in stereology are those related to the shape and size of particles. For example, the determination of the volume distribution of particles from measurements in planar sections. If only the variable v(volume) is considered, the problem is in general unsolvable because the probabilities P_0 and P_1 previously defined cannot be expressed in terms of v only. In order to have a well-defined problem that leads to an equation of type (1) it is necessary to consider enough variables x related to the shape and size that allow the definition of the probability functions $P_0(x)$ and $P_1(x,y)$. Such a set of x variables will be termed a complete set. The variable of interest, v, is some function of the variables x

$$v = v(x)$$

and its distribution function can be found after determining the distribution X(x) by solving eq.(1). Alternatively, the variable (or variables) of interest can be used as x variables, together with additional variables that define a complete set. The distribution of the variables of interest is then obtained by integration over the other x variables. These are then to be regarded as "auxiliary" variables.

The adequate choice of the variables x that form a complete set may require some additional information (or some assumption) on the shape of the particles. For example, if it is known or assumed that the particles are ellipsoidal, then three variables x are sufficient (e.g. the semi-axes of the ellipsoids or the volume and the two extreme semi-axes). For more complicated shapes more variables are required. We shall not discuss the problem further but admit that a finite number of such variables is sufficient, at least with a good degree of accuracy, to have a well-defined problem which is then formally solved by an equation of type (1).

SOLUTION OF INTEGRAL EQUATION FOR X(x)

The solution of eq.(1) to obtain X(x) is in general impossible by analytical methods. Various methods were given by Hilliard (1962) to solve the equations for oriented structures, but they are complicated and of no general applicability. These include a second differences method and methods based on Fourier or spherical harmonics expansions of X(x). Graphical methods for oriented structures were recently developed by Fortes (1990). The solution of a number of stereology problems related to size distribution of particles of a given, relatively simple shape, are also available in the literature (some of these are described in the book by Underwood, 1970).

In general, the solutions obtained by these methods are not stable, in the sense that errors in the measurement of the prints, i.e. on the measurement of Y(y), are amplified, leading to large errors in X(x).

The method that we propose to solve eq.(1) is based on the discretization of both distributions in classes. Eq.(1) is then transformed into a system of linear equations that can be solved by standard matrix methods. This discretization procedure is approximate, but is legitimate because in actual problems the experimental distribution Y(y) is obtained in the form of an histogram with intervals Δy of the variables y. The method is also adequate to predict the stability of the solutions and to prove that, for particular problems, the stability may indeed be intrinsically very poor, in the sense that any method of solution will lead to unstable solutions.

We first describe the method for one-variable problems, i.e., one variable x and one-variable y. Each interval, or class, of y, of amplitude Δy (which we assume for simplicity to be the same for all intervals) is identified by y_i, the central point of the interval. The experimentally determined histogram gives the "amount" Y_i of prints in the interval $y_i - \Delta y/2$, $y_i + \Delta y/2$

$$Y_i = \int_{y_i-\Delta y/2}^{y_i+\Delta y/2} Y(y)dy \tag{3}$$

It may be preferable to use, instead of Y_i, the average value Y_i^* of Y in the interval y_i, Δy_i:

$$Y_i^* = \frac{Y_i}{\Delta y} \tag{4}$$

Next we divide the interval of the variable x in sub-intervals of amplitude Δx, each interval having limits $x_k \pm \Delta x/2$. As will become apparent below, it is convenient to have the same number, n, of intervals of y and of x. Combining eqs.(1) and (3) we obtain

$$Y_i = \sum_k \int_{y_i-\Delta y/2}^{y_i+\Delta y/2} dy \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} X(x)p(x,y)dx \tag{5}$$

We approximate this equation by assuming X(x) to be constant in the interval $x_k \pm \Delta x/2$ with a value $X_k/\Delta x$, where X_k is the total amount of features in the interval. Then, defining the n^2 quantities

$$P_{ik} = \frac{1}{\Delta x} \int_{y_i-\Delta y/2}^{y_i+\Delta y/2} dy \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} p(x,y)dy = \bar{P}_{ik}\, \Delta y \tag{6}$$

where $\bar{P}_{ik}$ is the average value of p(x,y) in the intervals of x and y under consideration, we finally obtain

$$Y_i = \sum_k P_{ik} X_k \tag{7}$$

or, in terms of the average values in the intervals,

$$Y_i^* = \sum X_k^* P_{ik}^* \tag{8}$$

with

$$P_{ik}^* = P_{ik} \frac{\Delta x}{\Delta y} \tag{9}$$

The quantity P_{ik} can be interpreted as the contribution of a feature in the class x_k to the prints in the class y_i. Eq.(6) shows how this quantity can be calculated by averaging p(x,y). The approximation leading to the final eq.(7) contributes an error that tends to zero as Δx and Δy tend to zero.

Eq.(7) can be written in matrix form by introducing the square (n x n) matrix $P=(P_{ik})$ and column matrices Y and X with elements Y_i and X_k, respectively:

$$Y = P\,X \qquad P = (P_{ik}) \tag{10}$$

The solution is

$$X = P^{-1}\ Y \qquad P^{-1} = (P^{-1}_{ik}) \tag{11}$$

provided P is non-singular as we are assuming. The elements P_{ik} are of course all positive, but in general there will be negative P^{-1}_{ik}. Note that an interpretation of the P^{-1}_{ik} similar to that given above for the P_{ik} cannot be given, i.e., P^{-1}_{ik} cannot be interpreted in terms of a relation between classes of x and y.

Introducing line matrices $\tilde{x}$ and $\tilde{y}$ with elements x_i and y_k respectively, the average values of the two distributions are given by

$$\bar{y} = \tilde{y}\ Y$$

$$\bar{x} = \tilde{x}\ X = (\tilde{x}P^{-1})Y \tag{12}$$

The various moments of the two distributions can easily be calculated by equations of the same type, involving the line matrices $\tilde{y}_n$ and $\tilde{x}_n$, where n is an integer, and, for example, $\tilde{y}_n$ contains elements y^n_k.

If the numbers n_x and n_y of intervals of x and y, respectively, are not the same, the matrix P in eq.(10) is rectangular, instead of square, and the system of equations (10) for the unknown variables, X, will be impossible or undetermined, respectively, for $n_x > n_y$ and for $n_x > n_y$. It is therefore advantageous to take $n_x = n_y = n$, and this is always possible.

The generalization of the method of solution of eq.(1) to the case of more than one variable in X and/or in Y is straightforward. For example, for three variables x, the distribution X is defined by quantities $X_{ik\ell}$, defined as the "amount" of features with values of the variables x_1, x_2, x_3, respectively, in the intervals $x_{1i} \pm \frac{\Delta x_1}{2}$, $x_{2k} \pm \frac{\Delta x_2}{2}$, $x_{3\ell} \pm \frac{\Delta x_3}{2}$. Quantities $P_{ik\ldots rs\ldots}$ are introduced which are proportional to the average values of p(x,y) in the intervals ik... of y and rs... of x. These quantities have the same interpretation as before, i.e. they are proportional to the probability that a feature in the class rs... of x gives a print in the class ik... of y. The following equation replaces eq.7

$$Y_{ik\ldots} = \sum_{r,s\ldots} X_{rs\ldots} \cdot P_{ik\ldots rs\ldots} \tag{13}$$

This is a system of linear equations, which can be solved for $X_{\ell m \ldots}$. All that is required is to use a contraction procedure introducing column matrices Y and X and construct a matrix P with the $P_{ik\ldots rs\ldots}$. Eq.(13) then becomes formally identical to eq.(7). The number n_y of elements in Y is the product of the numbers of intervals used for the various variables y; and similarly for X. The matrix P is then a n_y x n_x matrix. For the reasons already explained, it is convenient to take $n_x = n_y$. The solution X is then obtained from the inverse matrix P^{-1} (cf. eq.(11)).

When auxiliarly variables x are introduced to have a complete set, the distribution of the x-variables of interest is obtained by summing all the X corresponding to each combination of intervals of those variables. It is easily shown that the required distribution can be obtained by forming a matrix, the elements of which are the sum of the elements of the matrix P corresponding to the intervals of the variables of interest, with no need to calculated to entire distribution of x-variables.

The method just describe was previously applied (Fortes and Ferro, 1988) to the determination of 3D dihedral angle distributions from angle measurements in 2D sections. This is a one-one variable problem (one variable x, one variable y). Another important example of a one-one variable problem will be dealt with in the following section: the determination of the length distribution of oriented lines in a plane by using various sets of test lines, each set having lines of fixed orientation.

An example of a problem with 2x2 variables is provided by dihedral angle distributions in a single phase polycrystal, where the smallest and largest dihedral angles are considered. This problem will be dealt with elsewhere.

DISTRIBUTION OF ORIENTED LINES IN A PLANE

The orientation of a line at a point is defined by α, the angle between the tangent to the line and a fixed direction in the plane ($0 < \alpha < \pi$). The length per unit area of lines with orientation $\alpha, d\alpha$ is $L_A(\alpha)d\alpha$. There is one variable x. The orientation of the test lines is defined by the angle, θ, between the normal to the test lines and the fixed direction ($0 < \theta < \pi$). This is the variable y. For a fixed orientation, θ, of the test lines, one counts the (average) number, per unit length, of points of intersection with the distribution of lines. This is denoted by $P_L(\theta)$. The two distributions are related by (Hilliard, 1962)

$$P_L(\theta) = \int_0^\pi L_A(\alpha)\ |\cos(\theta-\alpha)|d\alpha \tag{14}$$

This equation can be solved for $L_A(\alpha)$ using Fourier expansions of L_A and P_L (Hilliard, 1962). A graphical solution is also possible (Fortes, 1990). Using the method developed in this paper, we take $\Delta\theta=20°$ intervals of θ and $\Delta\alpha=20°$ intervals of α (i.e. the intervals are 0°-20°; 20°-40°; ··· 160°-180°). The quantities P^*_{ik} are, from eqs.(5) and (9):

$$P^*_{ik} = \frac{1}{\Delta\theta} \int_{\Delta\theta_i} \int_{\Delta\alpha_k} |\cos(\theta-\alpha)|d\theta d\alpha \tag{15}$$

where, for example, $\Delta\theta_i$ indicates the integration interval of θ. The integrals can be calculated analytically. The sum of the elements in a line or in a column of P^* is 2. The matrix P^* calculated for $\Delta\theta=\Delta\alpha=20°$ is given in Table 1. The inverse matrix P^{*-1} is given in Table 2; the sum of the elements in each line or column is 1/2. Both matrices are symmetrical relative to both diagonals.

TABLE 1

Matrix P^* for intervals of 20°

a	b	c	d	e	e	d	c	b
	a	b	c	d	e	e	d	c
		a	b	c	d	e	e	d
			a	b	c	d	e	e
				a	b	c	d	e
					a	b	c	d
						a	b	c
							a	b
								a

a = 0.3455
b = 0.3247
c = 0.2647
d = 0.1728
e = 0.0651

The matrix $Y=P_L$ contains the average numbers of points intersected per unit length of test lines of orientation θ_i=10°, 30°··· 170°. Each of these numbers can be obtained by using more straight lines than those for the orientation θ_i. For example, to obtain P_L for θ_i=10°, one can use test lines at 0°, 10°, 20° and take an average value. The distribution of orientations of the lines is then obtained from

$$L_A = P^{*-1}\ P_L \tag{16}$$

Each element of the matrix L_A gives the average length, per unit area and unit angle interval, of lines with orientation in the corresponding interval of α(i.e., 0--20°; 20°-40°, ···).

TABLE 2

Matrix P^{*-1} for intervals of 20°

-21.05	20.68	-20.16	19.21	-8.898	-9.387	19.71	-20.66	21.05
	-20.32	19.82	-19.32	18.57	-8.575	-9.537	19.85	-20.66
		-19.34	18.88	-18.63	18.31	-8.557	-9.537	19.71
			-18.42	18.21	-18.39	-18.31	-8.575	-9.387
				-18.00	18.21	-18.63	18.57	-8.898
					-18.42	18.88	-19.32	19.21
						-19.34	19.82	-20.16
							-20.32	20.68
								-21.05

Multiplication of an element of L_A by $\Delta\alpha=\pi/9$ gives the total length per unit area in the corresponding interval of α. Summing all these lengths we obtain the total length of lines per unit volume, which is, in virtue of a property of P^{*-1} referred to above, equal to the product of $\pi/2$ by the average value of the P_L for the various orientations. This is a well-known basic result of stereology (e.g. DeHoff and Rhines, 1968).

THE STABILITY OF THE SOLUTIONS

The experimental errors in the determination of the distribution Y(y) originate errors in the calculated distribution X(x) and it is important to assess the stability of the solution X(x) of eq.(1). In principle, this can be done by studying the properties of eq.(1) as regards the effect on X(x) of a perturbation in Y(y). Alternatively one may use the linearized forms (10) and (11) to evaluate the stability, by considering the errors in the X_i and Y_k. It should be pointed out that the stability of the solutions of eq.(1) is a property of this equation, independent of the method of solution used (if eventual approximations inherent to the method of solution are neglected).

The experimental errors in Y(y) (or in the Y_i) may be due to sampling or result from the measurement of the y variables. Two types of errors will be considered, namely constant (systematic) errors and random errors. We shall also refer to relative and absolute errors of X_i and Y_k, the latter being denoted by ΔX_i and ΔY_k , respectively.

If the relative error in Y_k is constant and equal to λ for all k, i.e., if $\Delta Y_k=\lambda Y_k$ where Y_k is the exact value, then obviously the relative error in X_i is also λ for all i.

The important situation to assess stability is that of random errors ΔY_k, which may be positive or negative. The repercussion of these errors in X_i will depend on the coefficients P^{-1}_{ik} for that ik and if the sign in all P^{-1}_{ki} is the same as (or opposite to) that in ΔY_k, a large amplifying effect in the error will occur. In order to quantify this amplifying effect we assume that the absolute value of ΔY_k is a constant ε. The maximum error, ΔX_i, that may result is then $\varepsilon \sum |P^{-1}_{ik}|$. If all Y_k had the same value, b, the corresponding valuek of X_i would be $b\sum_k P_{ik}$. We then introduce the quantity

$$E_i = \frac{|\sum_k P^{-1}_{ik}|}{\sum_k |P^{-1}_{ik}|} \tag{17}$$

which is inversely proportional to the absolute value of the maximum relative error in X_i for Y_k=constant and $\Delta Y_k=\pm\varepsilon$. The quantities E_i can vary in the interval [0,1] and the maximum error in X_i increases as E_i decreases.

The values of E_i for the matrix P^{*-1} of Table 2 are given in Table 3. The low values indicate a poor stability of the solution.

TABLE 3

Values of E_i for matrix P^{*-1} of Table 2 and errors $\Delta Y_k = \pm 0.1$*

i	E_i	ΔX_i +++++++++	ΔX_i +-+-+-+-+-+	ΔX_i -+-+--+-+
1-9	0.00311	0.0495	0.24	16.08
2-8	0.00318	0.0508	5.72	-7.89
3-7	0.00327	0.0496	-9.45	9.92
4-6	0.00336	0.0518	12.99	5.82
5	0.00341	0.0504	-14.66	1.80

* The successive signs of ΔY_k are indicated in each case.

E_i is a measure of the maximum relative error in the corresponding X_i, but the maximum error cannot, in general, occur simultaneously (i.e. for the same signs of $\Delta Y_i = \pm\varepsilon$) in all X_i.

Large errors do not occur in all X_i when the P^{-1}_{ik} are such that their sum in each column of P^{-1} is nearly the same in all columns. This is so because the distribution of positive and negative elements in the lines of P^{-1} will then be more "equilibrated" and not all lines of P^{-1} will be unfavourable in the arrangement of the signs relative to those of a given set ΔY_i. Therefore, the uniformity of the sum, $\sum_k P^{-1}_{ik}$, of the elements in the columns of P^{-1} is desirable for better stability. The matrix P^{*-1} of Table 2 satisfies this requirement since that sum is constant (equal to 1/2). Nevertheless, the stability of the solutions of the problem previously discussed in detail is rather poor, because of the low values of the E_i. The poor stability is evidenced in Table 3 in which the errors ΔX_i are calculated for $\Delta Y_k = \pm 0.1$ with various sequences of +0.1 and -0.1.

The two conditions ennunciated above for a good stability of the solutions were given in terms of the elements of P^{-1}. Conditions related to the elements of P can also be advanced which can be derived from those ennunciated for P^{-1}. A large value of E_i means that the sum of the elements of one sign in the i-line of P^{-1} greatly exceeds (in absolute value) the sum of the elements of the other sign. But P is the inverse matrix of P^{-1}, so that the product of a i-line of P^{-1} by the columns of P is zero, except for the i-column. As a consequence, and since all elements of P are non-negative, a large E_i means that the elements in the lines of P are far from being uniform, i.e., there are large variations between the different elements. Indeed, if the elements of a i-line of P are all equal, the values of E_k for $k \neq i$ are all zero. This condition of non-uniformity of the elements in the lines of P is not fulfilled by the matrix of Table 1.

Similarly, the uniformity of the sum of the elements in successive columns of P^{-1} implies uniformity in the sums of the lines of P. Indeed, if the sums of the columns of P^{-1} are all equal, the same happens with the sums of the columns of P, which is a general property of square matrices.

The uniformity of the sums of elements in the columns of P and the non-uniformity of the sums of elements in the lines of P, means that for a better stability the elements of P should be arranged in such a way that in each line or column large and small elements occur, but their sum should be roughly the same in all columns and in all lines. This in turns means that better stability of the X(x) solutions of eq.(1) occurs in those cases where p(x,y) changes appreciably with x and y, but in such a way that the integrals of p(x,y)dx and of p(x,y)dy over the entire intervals of x and y, respectively, vary little with y and x, respectively.

The stability of the solutions obtained by the discretization method can be improved by an appropriate choice of the intervals. In general, a reduction in the number of intervals improves the stability. For example: if the two first lines of the matrix P^{*-1} in Table 2 are replaced by the sum of the corresponding elements, which is equivalent to increasing the first interval of x from 20° to 40°, the value of E of the new line is 0.025, larger than those of E_1 and E_2 by a factor of ~ 8 (see Table 3). Similarly, it is possible to sum colums of P to improve the stability. But these improvements are obtained at the cost of a reduction in the quantity of information that is available.

CONCLUDING REMARKS

A general method of solution of the integral equation (1) was developed, which allows the determination of the distribution X(x) of variables, x, characteristic of three dimensional features, from the experimentally measured distribution Y(y) of variables, y, characteristic of the two-dimensional prints, obtained by intersection with test planes. The method is quite simple and the relevant matrix P^{-1} can be calculated for each problem.

While the method is general, the possibility of writting an equation of type (1) may require the introduction of more variables x, in addition to those that are of direct interest. The choice of these variables and the determination of the function p(x,y) in eq.(1) is an additional and sometimes complex problem that has not been discussed here. But the introduction of additional variables x is the only way that is left to actually solve some of the "unsolved" problems of stereology, such as the determination of the distribution of volumes of particles (or simply their average value) from measurements in planar sections. The additional variables x can be regarded as auxiliary variables in the sense that the distribution of the variables of interest can be calculated without determining the distribution of the auxiliary variabbles.

The solutions of some equations of stereology of type (1) are unavoidably unstable, and no method of solution can circumvent this characteristic. Only by improving the accuracy of the experimental determination of the distribution of the y variables, can the accuracy of the solutions be improved. The evaluation of the stability of the solutions can be done with the help of parameters calculated from the matrices used in the method developed in this paper.

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