ACTA STEREOL 1993; 12/2: 155-160
PROC 6ECS PRAGUE, 1993
ORIGINAL SCIENTIFIC PAPER

PROPERTIES OF THE VORONOI TESSELLATION CORRESPONDING TO THE GENERALIZED PLANAR GAUSS-POISSON PROCESS

Ivan Kohútek, Ivan Saxl*

Institute of Materials Research, Slovak Academy of Sciences,
Watsonova 47, 043 53 Košice, Slovakia
*Mathematical Institute, Academy of Sciences of the Czech Republic,
Žitná 25, 115 67 Praha 1, Czech Republic

ABSTRACT

The Voronoi mosaics corresponding to the planar Neyman-Scott process of point pairs and regular quadruples (vertices of a square) are investigated. The mean values of cell parameters are those of a Poisson-Voronoi tessellation (PVT) of the same intensity of generating points. If the inter-daughter distances are comparable with or smaller than the mean nearest neighbour distance of the parent process, then higher moments of cell area and perimeter distributions differ considerably from the PVT values. If, moreover, the orientation of clusters is fixed, then also a pronounced anisotropy of cell boundaries is observed. The results are compared with those of standard statistical quadrat testing methods and a good agreement is found.

Key words: Voronoi tessellation, planar Neyman-Scott process, Gauss-Poisson process, point clusters, statistical testing.

INTRODUCTION

The popularity of Voronoi tessellations corresponding to examined point patterns is based on the ability of human eye to perceive easier the departures from uniform randomnes, clustering, anisotropy, periodicity etc. by inspecting the induced Voronoi cells than by observing the point pattern itself. Moreover, it is rather tempting to consider the Voronoi tessellation as a natural dual representation of the underlying point pattern and to carry out a detailed statistical testing of its properties instead of testing the original point pattern, e.g. in order to reject or accept the hypothesis concerning the choice of a model for the investigated pattern.
The properties of the Voronoi tessellation induced by the Poisson point process (PPP) are well known from the theoretical (Meijering, 1953; Gilbert, 1962) as well as from numerous computational studies (e.g. Hinde and Miles, 1980; Miles and Maillardet, 1982). Voronoi tessellations induced by more general point processes are much less known.
The purpose of the present contribution is to compare the properties of Voronoi tessellation induced by the planar Gauss-Poisson process - GPP - (Stoyan *et al.*, 1987, 142-5) and its slightly formal generalization with those of the Poisson-Voronoi mosaic. GPP is a special case of the Neyman-Scott cluster process combining the poissonian randomness of the parent location with the regularity ot the daughter arrangement and its characteristics depend on the cluster size and orientation. Simplified versions of GPP have been examined, namely the representative cluster N_0 was

either a pair of points (process of pairs) or a quadruple of points forming vertices of a square (process of quadruples); the size of N_O has been varied and its orientation was either fixed or isotropic random.
The basic theoretical formulae for Poisson-Voronoi tessellation (PVT) are

$$E(A)=\lambda^{-1},\ E(S)=4/\sqrt{\lambda}=2L_A/\lambda,\ E(N)=6,\ E(A^2)=1.280\lambda^{-2}, \tag{1}$$

where A, S, N are the cell area, perimeter and number of cell edges, resp., and λ is the intensity of the underlying PPP (the first equality in (1) holds obviously for an arbitrary tessellation).
The sample characteristics of distributions estimated by Hinde and Miles (1980), namely variance σ, coefficient of variance CV, skewness β_1 and kurtosis β_2 are summarized in Tbl. 1.

Table 1. Characteristics of PVT

	A	S	N
σ	$0.529/\lambda$	$0.973/\sqrt{\lambda}$	1.335
CV	0.529	0.2433	0.223
β_1	1.033	0.193	0.432
β_2	4.599	2.983	0.206

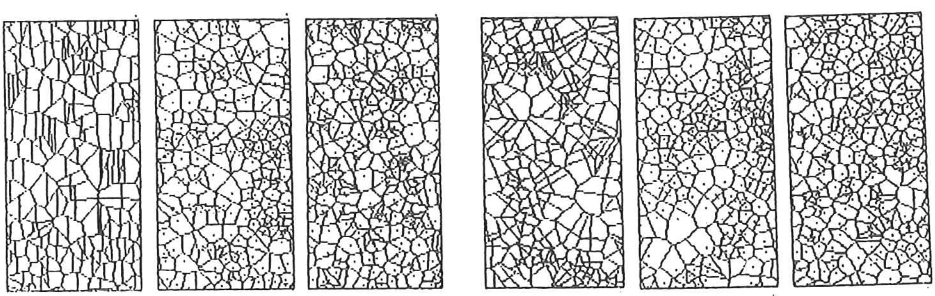
SIMULATIONS

Four types of GPP have been simulated by implanting a pair or regular quadruple of points into a point of the parent PPP of intensity $\lambda_p \equiv \lambda_X$:

A) point pairs of fixed orientation and of variable distance $2\xi_A \in [0.01,4]$ as measured in the units of $1/\sqrt{\lambda_A}$, which is twice the mean nearest neighbour distance $\sigma_p=0.5/\sqrt{\lambda_p}$ of the parent points,
B) as A) with the isotropic orientation of pairs,
C) regular quadruples of fixed orientation and of the variable edge length $2\xi_C \in [0.01,4]$ as measured in the units $1/\sqrt{\lambda_C}$,
D) as C with the isotropic orientation of quadruples.

The intensity λ_{cl} of the resulting cluster process has been the same in all cases and the mean number of points in the examined window was 1000. The parent intensities and cluster sizes at given value of ξ (in the units of $1/\sqrt{\lambda_p}$) have then been related by $\lambda_A=\lambda_B=2\lambda_C=2\lambda_D$ and $\xi_C=\xi_D=\sqrt{2}\xi_A=\sqrt{2}\xi_B$. The corresponding Voronoi tessellation was then constructed by the method described in Ferianc (1991). The edge effects have been removed by using a broad protecting frame. Several examples of simulated point patterns together with corresponding Voronoi mosaics are shown in Fig. 1.

RESULTS



A D

Fig. 1. Voronoi mosaics of GPP at the values 2ξ=0.1, 1 and 4 (in the units of $1/\sqrt{\lambda_p}$) of the cluster size for the cases A and D.

A. *Analysis of Voronoi mosaics.* All results relate to the normalized quantities $a=A\lambda_{cl}$, $s=S\sqrt{\lambda_{cl}}/4$ and N and are plotted in Fig's 2 - 4 as functions of the cluster size 2ξ (in the units of $1/\sqrt{\lambda_X}$, where X=A,B,C,D).

The mean values of cell area, perimeter and number of edges do not exhibit any systematic dependence on ξ and the sample averages have been $Ea=0.9995$, $Es=0.9992$ and $EN=5.9978$ with respect to the whole sample of $n = 2.8\times10^4$ examined cells thus lying within one standard deviation ($\sigma/\sqrt{n}=0.003$ for Ea, 0.0014 for Es and 0.008 for EN) of the PPP.

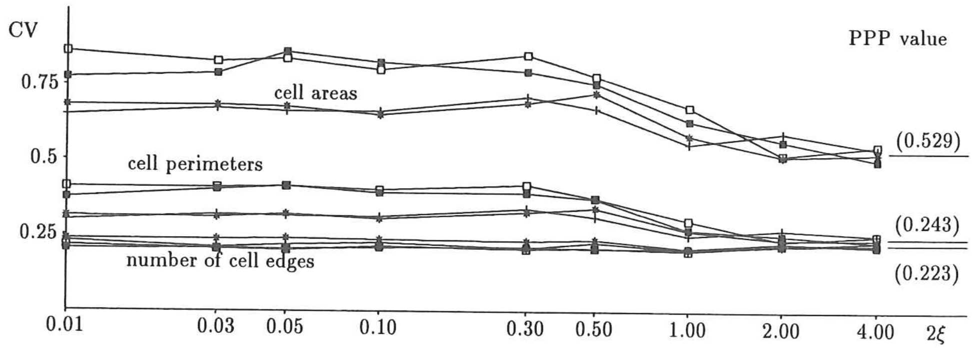


Fig. 2. The dependence of the coefficients of variance CV(a), CV(s) and CV(N) on the cluster size 2ξ. Notation: A(+), B(∗), C(□), D(■)).

On the other hand, some characteristics dependent on the higher moments of distributions reveal systematic dependence on cluster size in the range of $2\xi\in[0,2]$. The estimates of the second moments $\mu_2'(a)$ and $\mu_2'(s)$ based on 1000 cells seem to be sufficiently accurate and give smooth dependences of CV(a), CV(s) on ξ approaching the PPP values at $2\xi=4$ - compare Fig. 2. It seems also, that the difference between GPP and PPP mosaics is slightly greater in the cases A, C (fixed cluster orientation) than in the cases B, D (isotropic cluster orientation). In contrast to this behaviour, CV(N) is independent of cluster size - Fig. 2. The obligatory disappearence of triangles at small values of ξ ($2\xi\le0.3$) in cases C, D is compensated by a smaller frequency of polygons with high N. Even a more detailed inspection of the p.d.f. $f(N)$ did not show other differences in comparison with the Poisson-Voronoi tessellation or any dependence on either ξ or the type of cluster. Probably the size of the sample was insufficient in this respect.

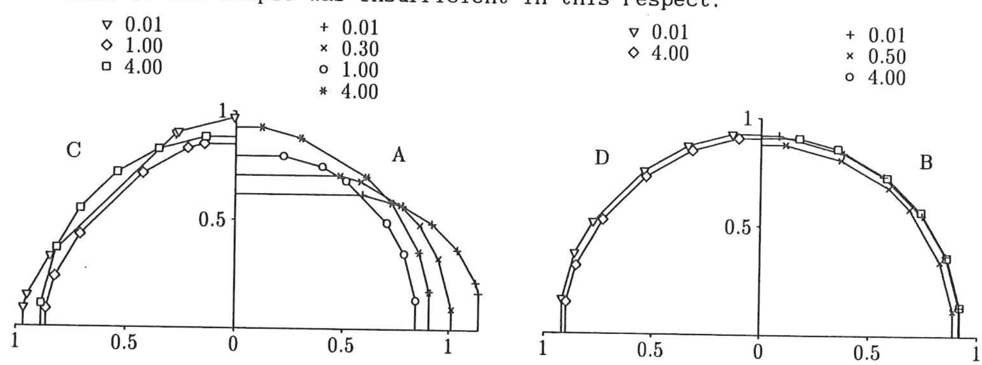


Fig. 3. Steiner compacts of Voronoi cells corresponding to clusters of fixed and isotropic random orientation at different values of cluster size 2ξ.

The values of skewness and kurtosis have not been determined with sufficient accuracy, nevertheles their scatter is high only in the region of small cluster sizes, whereas at $2\xi=4$ the PPP values are fairly matched.
The anisotropy of cell boundaries is quite pronounced at small values of ξ in the cases A and C as documented by the polygonal approximations of the corresponding Steiner compacts of mosaics (see e.g. Mecke *et al.*, 1990) obtained by the graphical method (Rataj & Saxl, 1992) - Fig. 3. The isotropy of cell boundaries is fully restored not sooner than at $2\xi=4$.

B. *Analysis of point patterns by quadrat methods.* For the comparison, also selected classical statistical methods for testing point patterns have been used. Leaving aside the distance method (for theoretical evaluation see Saxl, this issue) and second order methods, we have focused on the quadrat methods (Ripley, 1982; Stoyan *et al.*, 1987, 54-64; Cressie, 1991). They consist in the subdivision of the examined area into a number of smaller subareas $a(i)$, $i=1,2,\ldots,m$, and counting the numbers $n(i)$ of points falling into $a(i)$.
i) Index of dispersion ID. The quantity ID called the index of dispersion is defined by

$$ID = s^2(m-1)/\bar{n}, \tag{2}$$

where s^2 is the sample variance of $n(i)$ and $\bar{n}$ their sample mean. As the expected value En equals the variance for the Poisson distribution, $ID\simeq m-1$ and if, moreover, $m>6$ and $\bar{n}>1$, then ID follows approximately the χ^2-distribution with $m-1$ degrees of freedom. Consequently, we have chosen $m=6\div1000$. Assuming now that in any point of the parent PPP, a very small (with respect to the size of $a(i)$) cluster of p points is implanted, we obtain $ID\simeq p(m-1)$. Let $\alpha(m)$ be the edge length of the square $a(i)$ in the units of $1/\sqrt{\lambda}$ corresponding to the subdivision of the examined area into m subareas. Then we obtain for the chosen range of m the values $\alpha(m)=(9.1\div0.7)$ for point pairs (the cases A, B) and $\alpha(m)=(6.6\div0.5)$ for quadruples (the cases C, D). The results of the test method are shown in Fig. 4. Small clusters ($2\xi<0.5$) are correctly recognized within the whole range of m and the value of ID is approximately $p(m-1)$. Otherwise, the test breaks down whenever $\alpha(m)$ approaches the size 2ξ of the cluster.

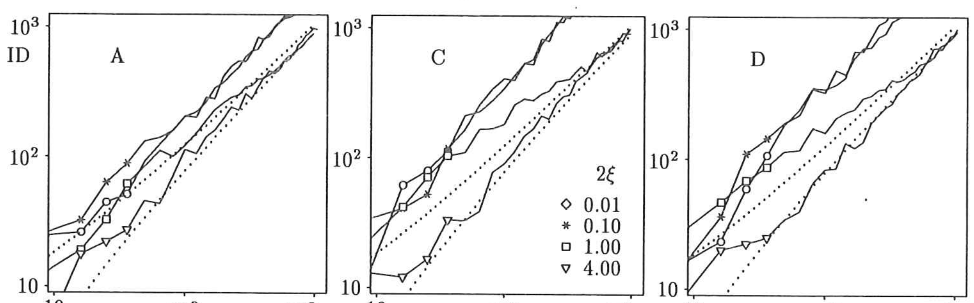


Fig. 4. The dependence of the index of dispersion on the number m of test quadrats for clusters of different size 2ξ and the double-sided confidence interval $[\chi^2_{m-1,1-\alpha}, \chi^2_{m-1,\alpha}]$ at the level $\alpha=0.05$ (dotted lines).

ii) Greig-Smith test for PPP. This test examines contiguous-quadrat data by a nested analysis of variance: setting $m=2^q$, we obtain a sequence of subdivisions m, $m/2$, $m/4,\ldots,2$ by joining subsequently two neigbouring regions into one rectangle. We then compare at different size levels i ($i=1$ corresponds to the smallest quadrat, the highest i to one half of the observed window) the sum of squares of number n of points in the blocs of i-th sub-division and in unions of neighbouring blocs (say $a(j)$, $a(k)$) merging into one in the $(i+1)$st subdivision . The evaluated statistic is $GS_i=\left[\sum_j n_j^2-0.5\sum_{[j,k]}(n_j+n_k)^2\right]/\bar{n}_i$, where $\bar{n}_i$ is the mean number of points in blocs of i-th subdivision (see Rataj *et al.*, 1993; Stoyan *et al.*, 1987, 54-64). Under Poisson hypothesis, it follows the χ^2-distribution with 2^{q-i} degrees of freedom. The value of q was selected so that $\bar{n}_1\cong 1$, namely $q=10$.

The results are summarized in Tbl. 2. A scatter of results uses to be considerable in Greig-Smith test, especially at small degrees of freedom. Consequently, the clusters of all sizes are not recognized at the size levels i=9-10. The presence of small clusters ($2\xi<1$) is revealed at all levels $i\leq 6$, large clusters with $2\xi\geq 1$ (note that the sizes of a quadruple and a bloc are comparable at $\{2\xi,i\}=\{4,7\}$, $\{2,5\}$, $\{1,3\}$, $\{0.5,1\}$) causes the rejection of Poisson hypothesis at one half (2ξ=1,2) or even only one quarter (2ξ=4) of situations irrespectively of the bloc size.

Table 2. Results of Greig-Smith tests

2ξ \ i	10	9	8	7	6	5	4	3	2	1	Total
.01	+* ■	+*□■	*	+* ■							11
.03	+* ■	+*□■	*	+	*						10
.05	*□	+*□	+	+*	*						9
.1	+*□	*□■	+*□■		*						11
.3	+*□■	+*□	*	□							9
.5	+	*□■	+*	□■						+*□■	12
1.0	+*□	+* ■	+*□■	+*□	+	*	*	*□■	*	*	21
2.0	*□	■	+*□	+ □	*□■	+*□	* ■	*□	+*□■	+* ■	25
4.0	+*□■	+*□■	*□■	+*□	*	*□	+* ■	+*□■	+*□■	+* ■	31

Notation: A(+), B(*), C(□), D(■).

The absence of a symbol at given values of i, ξ denotes that the Poisson hypothesis must be rejected at the level 95% for the cluster process represented by the missing symbol.

DISCUSSION

The above given results have been obtained with rather small samples of 1000 cells only, nevertheless the important differences between point patterns of GPP and PPP and between corresponding Voronoi mosaics have been revealed and the range in which they can be observed, namely $0<2\xi\leq 1$, or, equivalently, $0<\xi\leq\sigma_p$, has been determined. Note, that this range of values of 2ξ in terms of cluster process parameters is $0<\xi\leq\sqrt{2}\sigma_{cl}$ and $0<\xi\leq 2\sigma_{cl}$ for the cases A,B and C,D, respectively. Further, no observable difference between the examined cluster process and PPP was found for $\xi>2$, i.e. $\xi>4\sigma_p$.

For the comparison, the results of the recent paper by Rataj *et al.* (1993) can be mentioned. In this paper an oscillating point pattern, namely regular unit square lattice the points of which have been given an independent shift following a centred planar normal distribution with variance r^2I, has been examined by several methods of the statistical analysis. Considerable differences between oscillating pattern and PPP have been observed for $r\leq 1$, no differences were detected for $r\geq 4$, which is the same result as obtained here. Also the computation of the spherical contact distances (Saxl, 1993) gives a similar estimate of the critical range of the cluster size.

The comparison of mosaic analysis and quadrat count shows the close agreement between the both approaches. Nevertheless, when using the quadrat count, the results must be carefully analysed with respect to the mutual interference of cluster and quadrat sizes. Moreover, any interpretation of the the results going beyond simple "reject or accept" statement is difficult as the meaning of tested statistics is sensible only in the case of PPP. On the other hand, the mosaic analysis gives meaningful results concerning cell (i.e. zone of influence) area and shape distributions and/or their moments, which can be considered also as characteristics of the underlying point pattern.

It should be stressed, that the above results exhaust in no respect the possibilities of statistical testing based on Voronoi mosaics. First, we have omitted the investigation of interior cell angles θ described in detail by Hinde and Miles (1980) for PPP; increased frequency of $\theta=\pi/2$ at small values of ξ would be certainly observable in the cases C and D as well as departure from the mode and mean value positions. Moreover, the ordinary equilibrium state condition (Stoyan *et al.*, 1987, 264-8) or, equivalently, the normality condition (Moeller, 1989) $\mathbf{E}n_{01}=3$, i.e. the mean number of edges emanating from a vertex attains its lower bound, is heavily disturbed in the cases C and D (frequently four edges emanate from the vertex placed in the centre of a small quadruple). Consequently, the simple relations between intensities and marks of the three point processes derived from the mosaic (nodes, edge mid-points, cell centroids), which are valid for normal tessellations (Stoyan *et al.*, 1987, 264-8) do not hold either and the differences can be used for testing. Finally, also the edge length distribution can be examined and compared with that one of Poisson-Voronoi tessellation, which is known (Muche, 1993).

REFERENCES

Cressie NAC. Statistics for spatial data. New York: J Wiley & Sons, 1991: 588-97.

Ferianc P. Linear systolic array for Voronoi diagram construction in linear time. In: Geometrical Problems of Image Processing (Eds.: Eckardt U, Hübler A, Nagel W, Werner G). Research in informatics, Vol 4. Berlin: Akademie-Verlag, 1991:120-6.

Gilbert EN. Random subdivisions of space into crystals. Ann Math Stat 1962;33:958-72.

Hinde AL, Miles RE. Monte Carlo estimates of the distribution of the random Polygons of the Voronoi tessellation with respect to Poisson process. J Statist Comput Simul 1980;10:205-23.

Mecke J, Schneider RG, Stoyan D, Weil WRR. Stochastische Geometrie. Basel: Birkhäuser, 1990:85-120.

Meijering JL. Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. Philips Res Rep 1953;8:270-90.

Miles RE, Maillardet RJ. The basic structures of Voronoi and generalized Voronoi polygons. J Appl Prob 1982;19A:97-112.

Moeller J. Random tessellation in $\mathbb{R}^d$. Adv Appl Prob 1989;21:37-73.

Muche L.: Some new characteristics of the Poisson-Voronoi tessellation. (this volume)

Rataj J, Saxl I. Estimation of direction distribution of a planar fibre system. Acta Stereol 1992;11/Suppl.I:631-636.

Rataj J, Saxl I, Pelikán K. Convergence of randomly oscillating point patterns to the Poisson point process. Appl Math 1993;38,221-235.

Ripley BD. Spatial Statistics. New York: J Wiley & Sons, 1982:102-29.

Saxl I. Spherical contact distances in Neyman-Scott process of regular clusters. Acta Stereol (this volume).

Stoyan D, Kendall WS, Mecke J. Stochastic Geometry and Its Applications. Berlin: Akademie-Verlag, 1987.