ACTA STEREOL 1993; 12/2: 125-130
PROC 6ECS PRAGUE, 1993
ORIGINAL SCIENTIFIC PAPER

DISTRIBUTIONAL PROPERTIES OF THE POISSON-VORONOI TESSELLATION

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ABSTRACT

This paper presents formulas for the Voronoi tessellation which is generated by a stationary Poisson process in $\mathbb{R}^d$. Expressions are given for the chord length distribution and the edge length distribution function. Furthermore, the behaviour of the pair correlation function of the point process of vertices of the planar Voronoi tessellation for small arguments is discussed. Finally, some characteristics concerning the "typical" edge of the spatial Voronoi tessellation are given.

Key words: chord length, edge length, face angle, pair correlation function, Poisson-Voronoi tessellation, vertices.

1. INTRODUCTION

The Poisson-Voronoi tessellation has been studied intensively in many papers, (Gilbert, 1962; Miles, 1972; Miles, 1974; Miles, 1984; Møller, 1989) and books (Okabe, Boots, Sugihara, 1992; Stoyan, Kendall, Mecke, 1987.) But in spite of the simple structure of the model there are still many severe problems which are far away from a satisfactory analytical solution.

The present paper gives some results obtained for the Poisson-Voronoi tessellation. The determination of the chord length distribution functions is based on special geometrical properties of the Poisson-Voronoi tessellation and on the connection to the linear contact distribution function.

The chapters concerning the distributional properties of the edges and vertices of the Poisson-Voronoi tessellation are based on the Palm distribution of the point process of vertices, or, in other terms, of the neighbourhood of the "typical" vertex. [The word "typical" is used as in Stoyan, Kendall, Mecke (1987), p. 110.] There Formula (76) of Miles (1974) leads to analytical results of a form which is accessible for computations.

The determinations are long and complicated, so that this paper gives the main results only. Technical details are mostly omitted here and can be found in Muche (1993a), Muche (1993b) and Muche (1993c).

2. THE CHORD LENGTH DISTRIBUTION FUNCTION

The Poisson-Voronoi tessellation in $\mathbb{R}^d$ is defined with respect to the points of a stationary Poisson process Φ of intensity λ. With probability 1 there exists a unique cell Z_0 which contains the origin o. Z_0 is associated with that point z_0 of Φ which is the nearest neighbour of o i.e.

$$Z_0 = \left\{ y \in \mathbb{R}^d \middle| \|y - z_0\| \leq \|y - z\| \text{ for all } z \in \Phi \right\}. \tag{1}$$

The term contact distribution is used as in Stoyan, Kendall, Mecke (1987). For a given random closed set Ξ and a convex compact set B containing the origin o, the contact distribution function H_B is defined by

$$H_B(r) = P(\Xi \cap rB = \emptyset \mid o \notin \Xi), \quad r \geq 0. \tag{2}$$

Here Ξ is the union of all cell boundaries of the Poisson-Voronoi tessellation. Since Z_0 is convex and o is almost surely an interior point of Z_0 and the cells have disjoint topological interiors, it follows that $H_B(r) = 1 - P(rB \subset Z_0)$ for every B. If B is the unit segment $s(o,1)$, then the notation $H_l(r)$ is used; this function is said to be the linear contact distribution function. Geometrical considerations lead to the formula

$$H_l(r) = 1 - c_d \lambda \int_0^\infty \int_0^\pi \rho^{d-1} \sin^{d-2}\alpha \exp\left(-\lambda \nu_d(B_{r,\rho,\alpha})\right) d\alpha \, d\rho, \quad r \geq 0, \tag{3}$$

where $B_{r,\rho,\alpha}$ is the union of two spheres with radii ρ and $\sqrt{\rho^2 - \rho r \cos\alpha + r^2}$ and distance r between the midpoints and ν_d denotes the d-dimensional Lebesgue measure, c_d is a real constant and α is the angle between $s(o,1)$ and the line connecting o and the point with the polar coordinates ρ and σ. This result has already been found by Gilbert (1962) for the planar ($d = 2$) and the spatial ($d = 3$) case. It is wellknown, that the linear contact distribution function H_l and the chord length distribution function $L(r)$ are linked by

$$H_l(r) = \frac{1}{\bar{l}} \int_0^r \left(1 - L(l)\right) dl, \quad r \geq 0, \tag{4}$$

where $\bar{l}$ is the mean chord length, (Gilbert, 1962). Thus the chord length distribution function of the Poisson-Voronoi tessellation is

$$L(r) = 1 - \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \bar{l}\, \lambda^2 \int_0^\infty \int_0^\pi \rho^{d-1} \sin^{d-2}\alpha \left(\frac{\partial}{\partial r}\nu_d(B_{r,\rho,\alpha})\right) \exp\left(-\lambda \nu_d(B_{r,\rho,\alpha})\right) d\alpha \, d\rho, \quad r \geq 0. \tag{5}$$

The integral formulas for the distribution functions and for the corresponding density functions can be used for a numerical evaluation.

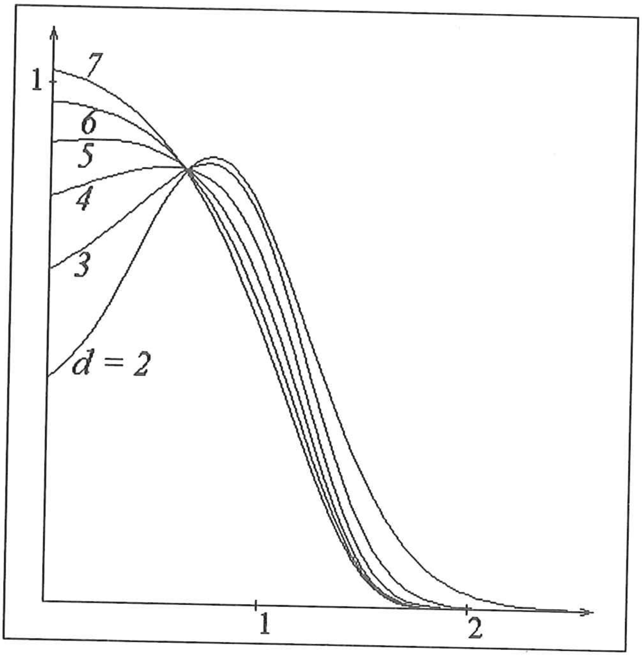


Fig. 1. The density function of the chord length distribution of the Poisson-Voronoi tessellation in $\mathbb{R}^d$, $d = 2,3,\ldots,7$. The intensity of the generating Poisson point process is 1.

3. EDGE LENGTH DISTRIBUTION FUNCTIONS

Consider the "typical" vertex of the Poisson-Voronoi tessellation in $\mathbb{R}^d$ which is surrounded by $d+1$ centres (points of Φ) $z_1, z_2, \ldots, z_{d+1}$ with the same distances

$$\|z_1\| = \|z_2\| = \ldots = \|z_{d+1}\| = \Delta. \tag{6}$$

Miles (1974) has given with Formula (76), p. 225

$$f_{\Delta,U_1,U_2,\ldots,U_{d+1}}(\delta,u_1,u_2,\ldots,u_{d+1}) \propto \delta^{d^2-1}\exp\left(-\nu_d(b(o,1))\delta^d\right)\cdot\nu_d(u_1,u_2,\ldots,u_{d+1}) \tag{7}$$

the joint distribution of the size of the sphere $b(o,\Delta)$ and the configuration of the centres onto it. The U_i $(i = 1,2,\ldots,d+1)$ are the projections of the z_i onto the unit sphere. In the case $d = 2$ several results are wellknown, i.e. the marginal density functions

$$f_\Delta(\delta) = 2\pi^2\lambda^2\,\delta^3\exp\left(-\lambda\pi\delta^2\right), \quad \delta \geq 0 \tag{8}$$

and

$$f_\Omega(\omega) = \frac{2}{3\pi}\sin\tfrac{\omega}{2}\left(\left(\pi - \tfrac{\omega}{2}\right)\cos\tfrac{\omega}{2} + \sin\tfrac{\omega}{2}\right), \quad 0 \leq \omega < 2\pi, \tag{9}$$

for one central angle Ω chosen at random i.e. that spanned by z_1 and z_2. Using these results, the distribution function of the length of the "typical" edge (edge length distribution function) of the planar Poisson-Voronoi tessellation can be written in the form

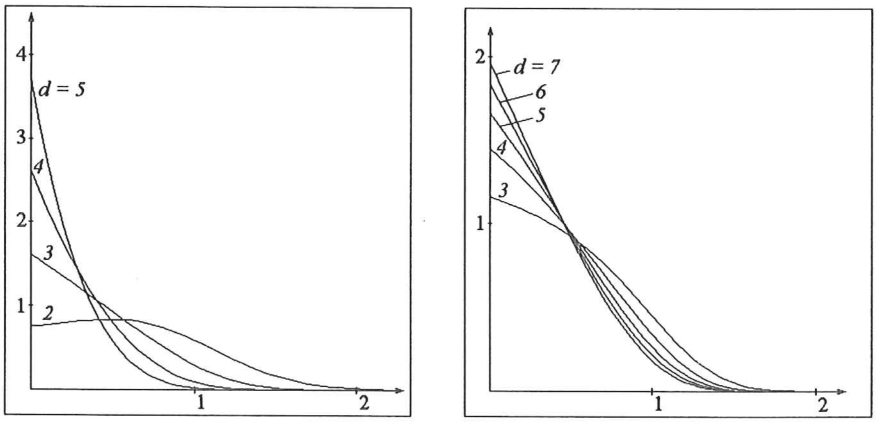
$$F_L(r) = \int_0^{\infty}\int_0^{2\pi} \Big(1 - \exp\big(- \lambda \nu_2(C(\delta,\omega,r))\big)\Big)\, f_{\Delta}(\delta)\, f_{\Omega}(\omega)\, \mathrm{d}\omega\, \mathrm{d}\delta,\ r \geq 0, \tag{10}$$

where $C(\delta,\omega,r)$ is the set

$$C(\delta,\omega,r) = b\Big(\underline{r}, \Big\| \underline{r} - \big(\delta, \tfrac{\omega}{2}\big)\Big\|\Big) \setminus b(o,\delta) \tag{11}$$

and $\underline{r}$ is the point with the coordinate r liing on the positive x_1-axis.

The method allows an extension in higher dimensional spaces, by partly solving Miles formula. Thus the edge length distribution function is obtained for the Poisson-Voronoi tessellation in $\mathbb{R}^d$ and for a plane intersection through a Poisson-Voronoi tessellation in $\mathbb{R}^d$, $d \geq 3$.



a) b)

Fig. 2. Density functions of the length of the "typical" edge (a) of the Poisson-Voronoi tessellation in $\mathbb{R}^d$ and (b) for a plane intersection through a Poisson-Voronoi tessellation in $\mathbb{R}^d$. The intensity of the generating Poisson point process is 1.

4. CHARACTERISTICS FOR THE VERTICES OF THE PLANAR POISSON-VORONOI TESSELLATION

The point process of vertices of the planar Voronoi tessellation, denoted by Φ^v is in close connection to the generating Poisson point process Φ in $\mathbb{R}^2$.

The K-function and the pair correlation function are important second order characteristics of a stationary and isotropic point process. Ripley's K-function or the reduced second moment measure function $K(r)$ (Stoyan, Kendall, Mecke, 1987, p. 50) for a point process in $\mathbb{R}^2$ is the mean value of points of it in a circle with radius r around its "typical" point, without the "typical" point itself, divided by the intensity of the point process. The corresponding pair correlation function $g(r)$ is defined as the second order product density (Stoyan, Kendall, Mecke, 1987, p. 47) divided by the square of the intensity of the point process. If $K(r)$ is differentiable in r the relation

$$g(r)=\frac{1}{2\pi r}\frac{\mathrm{d}}{\mathrm{d}r}K(r), \tag{12}$$

holds. Construction of special bounding functions $K_l(r)$ and $K_u(r)$ with

$$K_l(r)\le K(r)\le K_u(r), \quad r\ge 0 \tag{13}$$

with the properties $K_l(0)=K_u(0)=0$ and $\frac{\mathrm{d}}{\mathrm{d}r}K_l(0)=\frac{\mathrm{d}}{\mathrm{d}r}K_u(0)$ leads to the value of the derivative $\frac{\mathrm{d}}{\mathrm{d}r}K(0)=\frac{32}{9\pi\lambda^{\frac{1}{2}}}$. The appearance of a pole of the pair correlation function of a point process which is in close connection to the Poisson process may be surprising. But it is in correspondence with empirical results. Namely the pair correlation function of Φ^v has been studied in at least two papers by simulation, (Stoyan, Stoyan, 1990; Icke, van de Weygaert, 1991). In both papers great values of $g(r)$ were obtained for small r and thus a pole of an order of near 1 at $r=0$ was conjectured.

It can be proved that the pair correlation function takes a pole of exact first order at $r=0$.

5. CHARACTERISTICS OF THE SPATIAL POISSON-VORONOI TESSELLATION

In counterpart to the planar case, Miles' Formula (76) was hardly applied for the spatial case until now. In a long but elementary way of rotations and other transformations Miles' Formula takes a form which is more tractable for the analytical determination of some characteristics concerning the "typical" edge.

The "typical" edge of the Poisson-Voronoi tessellation in $\mathbb{R}^3$ has almost surely three emanating faces which generate three random angles (face angles) Θ_1, Θ_2 and $\Theta_3=2\pi-\Theta_1-\Theta_2$ perpendicular on the "typical" edge. Their common density is

$$f_{\Theta_1,\Theta_2}(\vartheta_1,\vartheta_2)=\frac{2^6}{3\pi^2}\sin^2\vartheta_1\sin^2\vartheta_2\sin^2(\vartheta_1+\vartheta_2), \quad \pi-\vartheta_1\le\vartheta_2<\pi, \quad 0\le\vartheta_1<\pi \tag{14}$$

and the marginal density of one of the face angles Θ is

$$f_\Theta(\vartheta)=\frac{8}{3\pi^2}\sin^2\vartheta\Big[\vartheta(3-2\sin^2\vartheta)-3\sin\vartheta\cos\vartheta\Big], \quad 0\le\vartheta\le\pi \tag{15}$$

with $E\Theta=\frac{2}{3}\pi$ and $\mathrm{var}\Theta=\frac{\pi^2}{18}-\frac{3}{8}$.

Consider now the circle perpendicular to the "typical" edge spanned by its three neighbouring centres. If x denotes one point of this circle and e_1 and e_2 the endpoints of the "typical" edge, then a random angle Γ is formed by x,e_1 and e_2. Its density function is

$$f_\Gamma(\gamma)=\frac{105}{128}(1+\cos\gamma)^2\sin^5\gamma\ , \quad 0\le\gamma\le\pi, \tag{16}$$

with $E\Gamma=\frac{849}{2^{11}}\pi$ and $\mathrm{var}\Gamma=\frac{1017951}{2^{22}}\pi^2-\frac{50089}{22050}$.

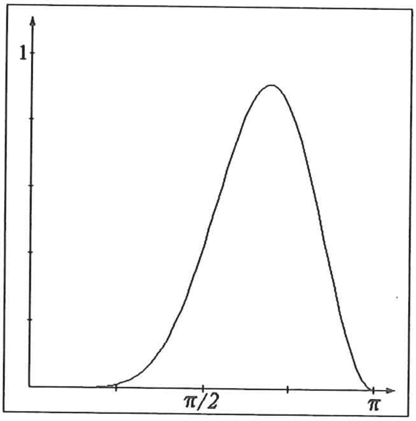


Fig. 3. Density function of the face angle, perpendicular to the "typical" edge, spanned by two of the emanating faces.

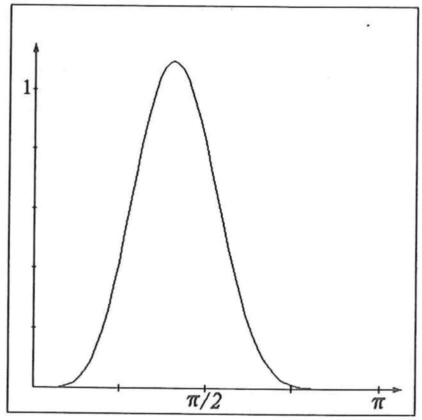


Fig. 4. Density function of the angle spanned by the line passing one neighbouring centre and one endpoint of the "typical" edge and the "typical" edge.

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