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A REMARK ON THE AREA ORIENTATION DISTRIBUTION

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ABSTRACT

This paper discusses the concept of area orientation distribution. It gives the exact area orientation distribution for rectangular and elliptic particles. The behaviour of these distributions confirms the assumption that area orientation distributions show a blurring effect around the expected orientation, which depends on the particle shape. Nevertheless, the area orientation method seems to give correct estimates of the main orientation of particle systems.

Key words : area orientation distribution, rectangle, ellipse.

INTRODUCTION

The investigation of the orientation distribution of particle systems is an important problem for biologists, material scientists and other researchers. For the definition of the direction of a particle or of a particle system there are various concepts. Some of them are based on the orientation of the particle's boundary and are thus not very attractive. A natural concept is the use of the direction of the maximum Feret diameter of the particle, which supplies satisfactory results for well-formed particles, such as ellipses. But for some special cases unnatural results are obtained by this method. For example, for a system of equal rectangles orientated all in the same direction, the orientation distribution would be a degenerated distribution concentrated on the two directions of the diagonals of the rectangles. This contradicts greatly the subjective impression, which expects a direction parallel to the longest rectangle edge.

Another concept, the so called *area orientation* of particles, suggested by Odgaard, Jensen and Gundersen (1990), seems to be more suitable. This area orientation is given by the distribution of the direction the chord of maximum length passing a randomly chosen inner point of a particle. Unfortunately, in the special case of ellipses we obtain a blurring effect around the orientation of the longer axes. The analogous effect for rectangles has been already mentioned by Stoyan and Beneš (1991). In the following the corresponding area distribution density is calculated exactly which has an interesting form. Simulations have led to the area orientation distribution for ellipses given in the last section of the paper.

DETERMINATION OF THE LONGEST CHORD INSIDE OF A RECTANGLE

Consider a rectangle $ABCD$ with fixed side lengths a and b, with $a \geq b$ in a Cartesian coordinate system, where A lies in the origin, B on the positive x-axis and D on the positive y-axis, with $\overline{AB} = a$ and $\overline{AD} = b$. A point $Z = (x,y)$ is chosen randomly in the interior of $ABCD$, with x and y uniform on $[0,a]$ and $[0,b]$. Then there exists a chord of maximum length in the set of all chords which are running through Z. With probability one this chord is unique. It forms an angle φ with the x-axis, $0 \leq \varphi \leq \frac{\pi}{2}$.

Because of the symmetry of the problem it is sufficient to consider it in one quarter of $ABCD$ only, e.g. in the rectangle with vertices C,F,M, and E, see Fig. 1. It is easy to see that the longest chord belonging to an arbitrarily choosen point Z is starting in a vertex of the rectangle $ABCD$. For the case of Z in the interior of the subrectangle $CFME$ only chords starting in A or C are of interest, because they are always longer than those starting in B or D. If Z is a point in the triangle ΔCME, then the chord starting in A is the longest one.

More complicated is the case if Z lies within ΔCFM. Then the longest chord is starting in C, if the distance $\overline{CZ}$ is sufficiently long and otherwise it is starting in A. Geometrical considerations allow a determination of the border line between these two domains corresponding to A and C.

Let be H a point of this border line. Then both chords passing H starting in A and C have the same length. The distance $\overline{AH} = l_1(\alpha_1)$, using α_1 for the angle $\sphericalangle BAH$, can be written as

$$l_1(\alpha_1) = a\,\frac{b\sin\alpha_1 - \sqrt{b^2 - a^2\sin^2\alpha_1}}{a\sin^2\alpha_1 - \cos\alpha_1\sqrt{b^2 - a^2\sin^2\alpha_1}}\,, \quad \arctan\tfrac{b}{a} \leq \alpha_1 \leq \arctan\tfrac{2b}{a}\,. \tag{1}$$

Analogously the distance $\overline{CH} = l_2(\alpha_2)$ satisfies

$$l_2(\alpha_2) = b\,\frac{a\cos\alpha_2 - \sqrt{a^2 - b^2\cos^2\alpha_2}}{b\cos^2\alpha_2 - \sin\alpha_2\sqrt{a^2 - b^2\cos^2\alpha_2}}\,, \quad 0 \leq \alpha_2 \leq \arctan\tfrac{b}{a}\,, \tag{2}$$

where α_2 denotes the angle $\sphericalangle FCH$, see Fig.1. For an inner point Z of ΔCFM the longest chord is starting in C if and only if Z belongs to the sector bounded by $\arctan\frac{b}{a} \leq \alpha_1 \leq \arctan\frac{2b}{a}$ and the curve described by (1). If Z is a point outside of this sector, then the longest chord passing Z is that which starts in A.

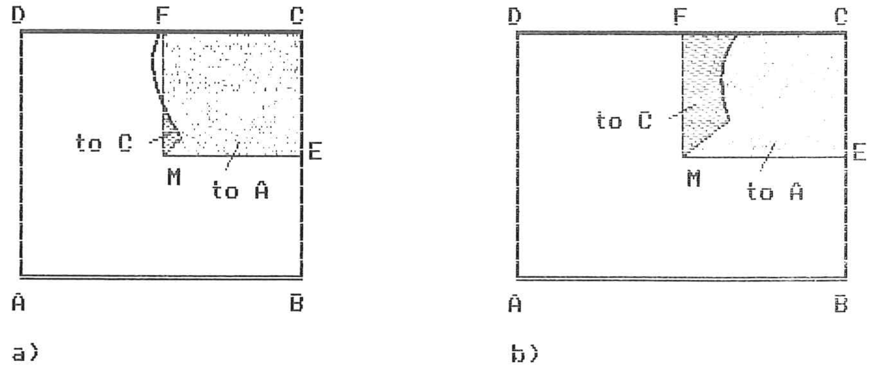

Fig. 1. Subdivision of the rectangle into regions, where an inner point Z connected with A or C generates the longest chord. The border line intersects the straight line limited by F and M, (a). Otherwise, it lies completely inside of the subrectangle $CFME$ for a sufficiently great ratio $\frac{a}{b}$, (b).

DETERMINATION OF THE AREA DISTRIBUTION FUNCTION FOR A RECTANGLE

Using the results of the previous section it is possible to determine the probability that the random angle φ formed by the side AB and the longest chord generated by a random point Z is less than a given value α. Because of the uniformity assumption, this probability $P(\varphi < \alpha)$ equals the quotient of the area of the set of all points which generate a maximum chord with an angle less than α, divided by the rectangle area. Consequently, with (1) and (2)

$$P(\varphi<\alpha)=\begin{cases} \frac{2}{ab}\int\limits_0^{\alpha}\left[\max\left(0,\left(\frac{a}{2\cos\alpha_2}\right)^2-l_2^2(\alpha_2)\right)\right]d\alpha_2\,, & 0\le\alpha<\arctan\frac{b}{2a},\\ \frac{2}{ab}\int\limits_0^{\alpha}\left[\max\left(0,\left(\frac{a}{2\cos\alpha_2}\right)^2-l_2^2(\alpha_2)\right)\right]d\alpha_2+\frac{(2a\tan\alpha-b)^2}{2ab\tan\alpha}\,, & \arctan\frac{b}{2a}\le\alpha<\arctan\frac{b}{a},\\ \frac{1}{2}+\frac{2}{ab}\int\limits_0^{\arctan\frac{b}{a}}\left[\max\left(0,\left(\frac{a}{2\cos\alpha_2}\right)^2-l_2^2(\alpha_2)\right)\right]d\alpha_2+ & \\ +\frac{2}{ab}\int\limits_{\arctan\frac{b}{a}}^{\alpha}\left[\left(\frac{b}{\sin\alpha_1}\right)^2-\max\left(\left(\frac{a}{2\cos\alpha_1}\right)^2,\,l_1^2(\alpha_1)\right)\right]d\alpha_1\,, & \arctan\frac{b}{a}\le\alpha<\min\left(\arcsin\frac{b}{a},\,\arctan\frac{2b}{a}\right),\\ 1, & \min\left(\arcsin\frac{b}{a},\,\arctan\frac{2b}{a}\right)\le\alpha. \end{cases}$$

The density function of the angle φ is very complicated. The exact determination of it requires to solve fourth-degree polynomials in α, the coefficients of which are depending on the ratio $q = \frac{a}{b}$. It is necessary to distinct the cases $1 \leq q < \frac{2}{\sqrt{3}}$, $\frac{2}{\sqrt{3}} \leq q < \frac{3\sqrt{3}}{2\sqrt{5}}$, $\frac{3\sqrt{3}}{2\sqrt{5}} \leq q < \sqrt{6\sqrt{3}-9}$, $\sqrt{6\sqrt{3}-9} \leq q$.

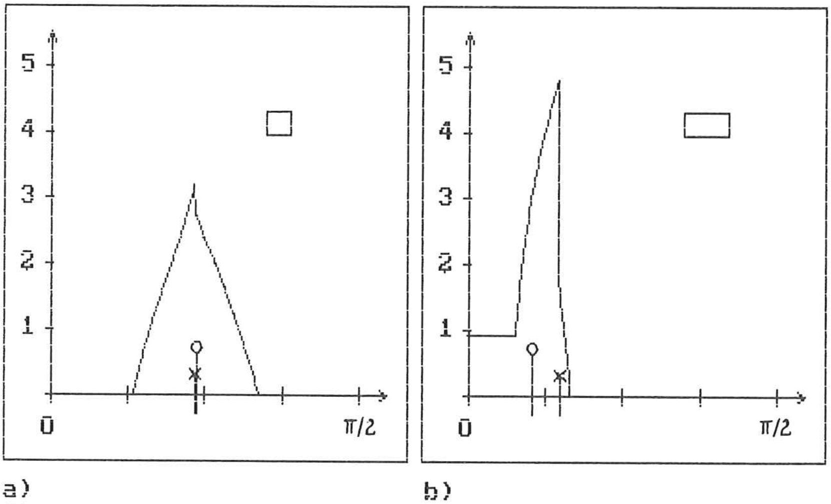

Fig. 2. The density function of the angle φ generated by the longer side of the rectangle and the maximum chord passing a uniform point Z for different ratios $q = \frac{a}{b}$, $q = 1.1$ (a) and $q = 2$ (b). The direction of the diagonal is marked by $\times$ and the expectation of φ by o.

The interval $\frac{2}{\sqrt{3}} \leq \frac{a}{b} < \sqrt{6\sqrt{3}-9}$ is most interesting. Here the density of the angle φ is positive in a small neighbourhood of the origin and equal zero in an interval of positive length behind them. Examples of such density functions are shown in Fig. 2. a and b. The curve in Fig. 2.b is plotted for a ratio $\frac{a}{b} = 2$. It is quite similar to Fig.3 in Stoyan and Beneš (1991). Clearly, if a sign is given to the angles, then their distribution is symmetric to the "main direction".

THE AREA ORIENTATION DISTRIBUTION FOR AN ELLIPSE

Consider an ellipse of half-axis lengths a and b with $a \geq b$. Again, for a uniform point Z inside of this ellipse there exists a unique longest chord. The determination of the density function of the angle φ between this chord and the longer half-axis leads to a fourth-degree polynomial depending on three parameters. The formulae are still more complicated than in the rectangular case, see Fig. 3.

Fig. 4 shows some estimated density functions of the angle φ for different ratios a:b, which are obtained by simulation. For this purpose points with coordinates uniform to the ellipse have been thrown onto it. To every of this points an approximate value for the maximum chord length and its angle φ was computed by an iteration procedure. The density function of the angle φ was estimated basing on this.

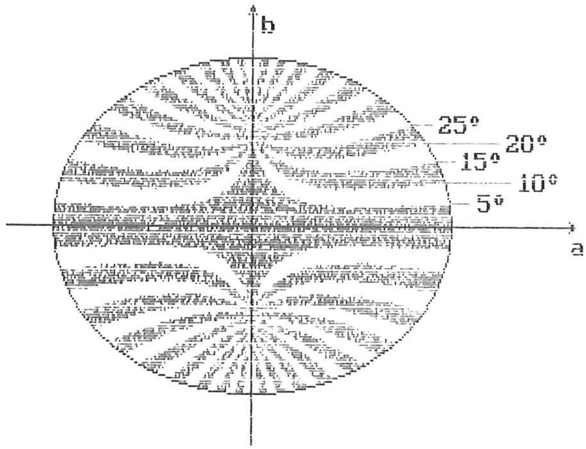

Fig. 3. Intersection of an ellipse into regions, where the angle φ between the maximum chord and the longer half-axis has values between given limits (differences of 5 degrees, a:b = 1.2).

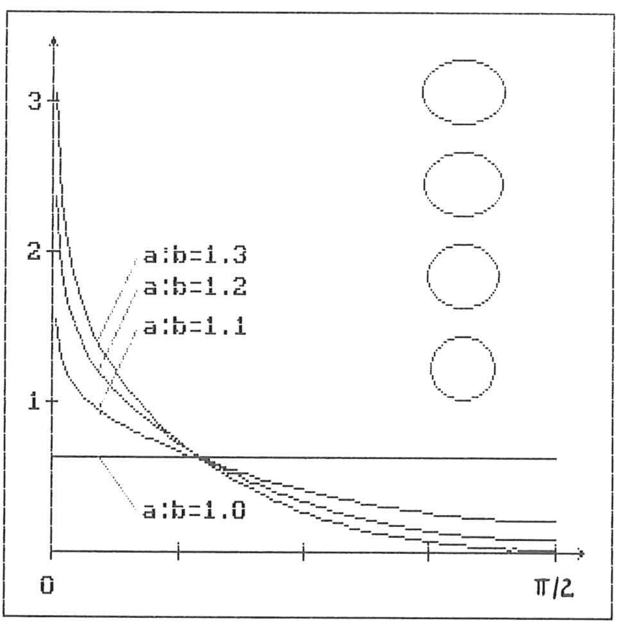

Fig. 4. Density function of the angle φ between maximum chord and longer half-axis for different ratios a:b.

In contrast to the rectangle, this density function has a maximum value at $\alpha = 0$. This fact follows from the smoothness of the boundary of the ellipse. The area orientation density functions for ellipses depend on the elongation of the ellipses. The more elongated the ellipse, the more concentrated is the density function around the direction of the longest half-axis. Clearly, for circles a uniform distribution is obtained.

DISCUSSION

For the particles considered the area orientation distribution shows remarkable deviations of the angles from the expected main direction. However, since the deviations are symmetric with respect to this direction, the mean of a sample of a angles obtained by the area orientation method is an unbiased estimator of the main direction. Also for particles of different shape and orientation it can be expected that the area orientation method gives reasonable results for the main direction.

The investigation of the variability of the particle orientation seem to be much more difficult. Probably complicated deconvolution procedures are necessary in order to extract useful information from empirical area orientation distributions. Perhaps, the results of this paper may help in the interpretation of such distributions.

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