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# USE OF THE DISECTOR TO ESTIMATE THE EULER CHARACTERISTIC OF THREE DIMENSIONAL MICROSTRUCTURES

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## ABSTRACT

The net volume tangent count may be performed on a disector sample of a microstructure; this count provides an unbiased estimate of the Euler characteristic of the feature set analyzed. For simply connected features (not necessarily convex), the Euler characteristic is equal to the number of separate parts of the feature. If the structure consists of a single multiply connected network, the Euler characteristic is the negative of the connectivity of the network.

connectivity, disector, number, tangent count, topology

## INTRODUCTION

Sterio (1984) has introduced the device called the **disector** as a sampling probe that may be used to estimate the number of features in a three dimensional microstructure. This paper presents a generalization of this procedure which permits a straight-forward estimation of a more general topological property of a microstructure: its Euler characteristic (Santalo, 1967). The method is based upon the application of the **volume tangent count** (DeHoff and Rhines, 1968) to the structure.

### Theoretical Background

The two topological properties of interest in the characterization of three dimensional microstructures are the **number** of disconnected parts in a feature set and the collective **connectivity** of all features in the set. The connectivity is the number of redundant connections in the skeleton that represents the feature set, and may be visualized as the number of times one may cut through the members of the set without increasing the number of separate parts. Features without redundant connections are called **simply connected** and have connectivity equal to zero. A feature that forms a network may have a very large connectivity. These two properties are usually reported as values per unit volume of microstructure and are designated by the symbols $N_V$ and $C_V$.

The **Euler characteristic** of the feature set is defined to be the difference, $(N_V - C_V)$. This quantity is a useful combination of these two topological properties because it is simply related to the **spherical image** (Santalo, 1967) of the feature set.

## The Spherical Image of a Feature

The spherical image of a feature is a map of its surface normals on the sphere of orientation. To visualize the spherical image, consider an element of surface on the feature shown in Figure 1a. If the surface is smooth, then any point P has a unique tangent plane. The normal to the surface at P is a unit vector that is perpendicular to the tangent plane at P. The spherical image of the surface at P is obtained by translating the normal vector to the center of a unit sphere, Figure 1b. The tip of the vector identifies a point, P', on the unit sphere which is the spherical image of P on the surface.

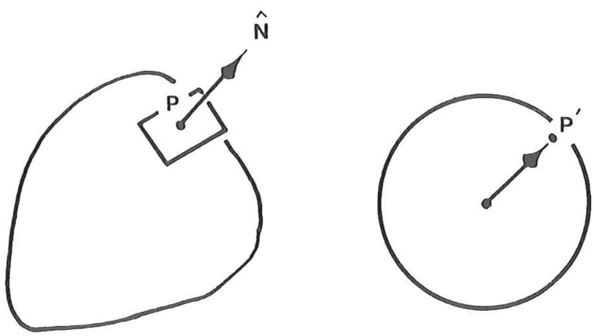


Figure 1. A point P on a smooth surface has a unique tangent plane and surface normal (a); P' is the spherical image of P (b).

The spherical image of a finite segment of smoothly curved surface is a segment of area on the unit sphere. It is evident that, for a surface bounding a convex body, the spherical image covers the unit sphere exactly once, Figure 2; thus, the spherical image of every convex body is equal to the area of the unit sphere: $4\pi$. If the bounding surface of a convex body is not everywhere smooth, i.e., if it has edges and corners, its spherical image remains $4\pi$ if the contributions of the edges and corners are properly visualized, Figure 3. The spherical image of a point P on an edge, Figure 3b, is the segment of the great circle lying between the spherical images of the normals at P to the surfaces that meet to form the edge. The spherical image of a corner, Figure 3c, is a segment of area on the unit sphere.

It can be shown that, for **non-convex** simply connected bodies, the **net** spherical image of the bounding surface remains $4\pi$ (Santalo, 1967). Any departure from convexity will be accompanied by the presence of some saddle surface, i.e., surface for which the two principal radii of curvature are of opposite signs. Let the mapping of the normals to saddle surface be

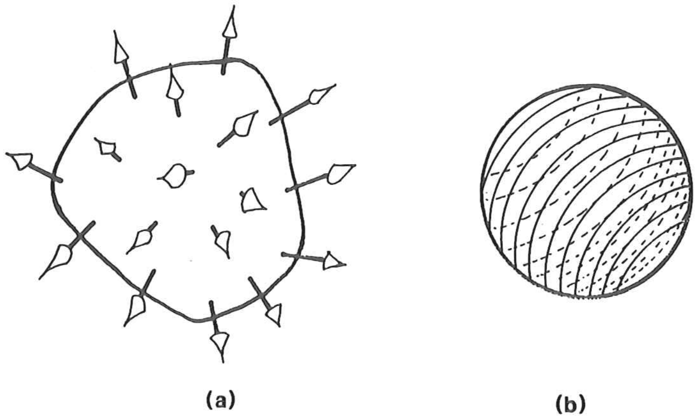


Figure 2. The set of normals for a smooth, closed convex surface (a) map as a set of spherical images that cover the unit sphere of orientation exactly once (b).

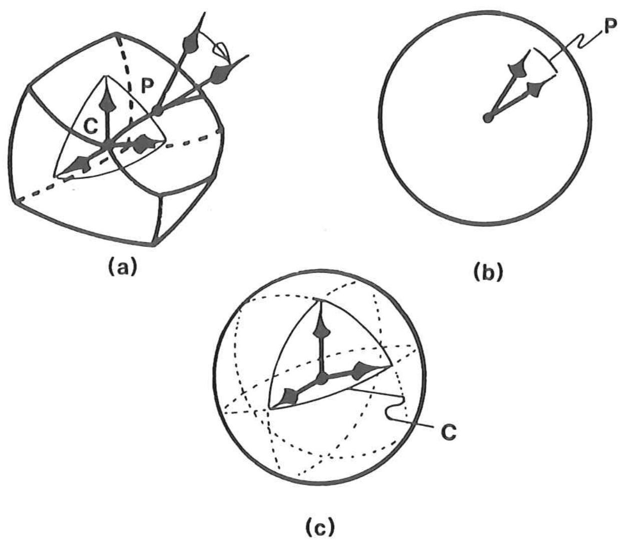


Figure 3. The spherical image of a point P on an edge of a polyhedron (a) maps as a segment of a great circle (b). The spherical image of a corner of a polyhedron (a) is a spherical polygon (c).

assigned a **negative** spherical image, $\Omega_{+-}$, while that for convex ($\Omega_{++}$) and concave ($\Omega_{--}$) elements is defined to be positive. Then the spherical image of saddle surface will exactly balance that contributed by concave surface elements plus excess convex surface elements, so that

$$\Omega_{net} = \Omega_{++} + \Omega_{--} - \Omega_{+-} = 4\pi \qquad (1)$$

Thus, the net spherical image of the surface bounding any simply connected body is a topological invariant and is equal to $4\pi$.

This theorem generalizes in a straight-forward manner to encompass multiply connected bodies. For the surface bounding a three dimensional feature with connectivity C, the net spherical image can be shown to be:

$$\Omega_{net} = 4\pi\ (1-C) \qquad (2)$$

Some plausibility to this theorem may be derived by inspection of Figure 4. The outside of the torus is everywhere convex and has surface normals that cover the unit sphere exactly once. The inside of the torus is composed of saddle surface elements; its normals also cover the unit sphere exactly once, and this image is assigned a negative sign. Thus, the net spherical image is zero. Each additional hole through the body contributes an additional $-4\pi$ to the net spherical image, yielding equation (2).

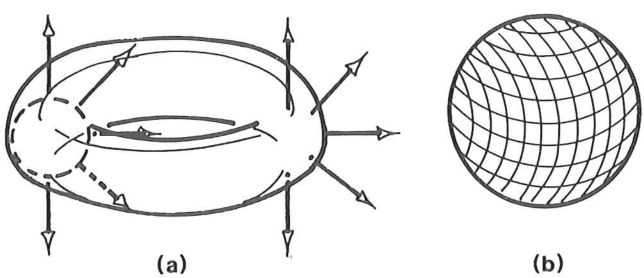


Figure 4. Surface normals along a "meridian" on the outside convex surface of a torus (a) map as half a meridian on the unit sphere (b). Surface normals along a "meridian" on the inside of a torus also map as half a meridian on the unit sphere (b) but are assigned a negative value. Rotation of these mappings about the axis of revolution of the torus covers the unit sphere exactly once for the outside convex surface, and exactly once, but with a negative sign, for the inside. Thus, the net spherical image of a torus is zero.

## The Volume Tangent Count

Consider a microstructure composed of a collection of three dimensional features. Imagine that this three dimensional structure is sampled by sweeping a **plane** through the volume of the structure and noting tangents formed by this plane with elements of the surface bounding features. Separate counts are made of the number of tangents formed with convex (++), concave (--) and saddle (+-) surface elements. If the boundaries of the feature set are not smooth, then tangents formed with edges and corners are included in each category. The net volume tangent count is defined to be:

$$T_{Vnet} = T_{V++} + T_{V--} - T_{V+-} \tag{3}$$

where $T_{Vij}$ is the ratio of the number of tangents formed in category ij to the volume swept out by the plane in performing the analysis.

The tangent count in each category measures the spherical image of surface in that category (DeHoff and Rhines, 1968). Thus,

$$T_{V++} = \Omega_{V++}/2\pi \tag{4a}$$

$$T_{V--} = \Omega_{V--}/2\pi \tag{4b}$$

$$T_{V+-} = \Omega_{V+-}/2\pi \tag{4c}$$

and

$$T_{Vnet} = \Omega_{Vnet}/2\pi \tag{4d}$$

According to equation (2), the net spherical image of a feature is a topological invariant, related to the connectivity of the feature. For a collection of features, the net spherical image per unit volume is thus

$$\Omega_{Vnet} = 4\pi\ (N_V - C_V) \tag{5}$$

Combining equations (4d) and (5) yields a simple relation between the net volume tangent count and the Euler characteristic per unit volume of microstructure:

$$T_{Vnet} = 2\ (N_V - C_V) \tag{6}$$

## Application to the Disector

The disector (Sterio, 1984) is a rudimentary serial section sample of a microstructure. It consists of two planes cut through the sample a small distance apart. The spacing is small enough so that inferences can be drawn about associations of features appearing upon one plane with those appearing on the other. It is possible to use a disector as a basis for performing the volume tangent count. It is necessary to assume that the disectors included in a given analysis provide an isotropic uniform random (IUR) sample of the structure. The volume sampled by the sweeping plane is the volume contained within the disector; the direction of sweep of the plane is normal to the planes bounding the disector. A feature-by-feature comparison on the two sections permits inference of the tangents formed with surface elements within the volume of the disector in each of the categories of interest:

a. A tangent with convex surface elements ($T_{V++}$) is inferred if a feature appears on the second plane that cannot be assigned as a continuation from a feature on the first plane. A count in this category is also inferred if a feature on the first section disappears on the second section.

b. A tangent count for concave surface elements ($T_{V--}$) is assigned if an isolated segment of the matrix (not-feature) appears on the second section that is not a continuation from the first plane, or if an isolated segment of matrix disappears between sections.

c. A tangent count is assigned to saddle elements ($T_{V+-}$) for each branching and joining event inferred to occur between the sections.

Examples of each of these counts are shown in Figure 5.

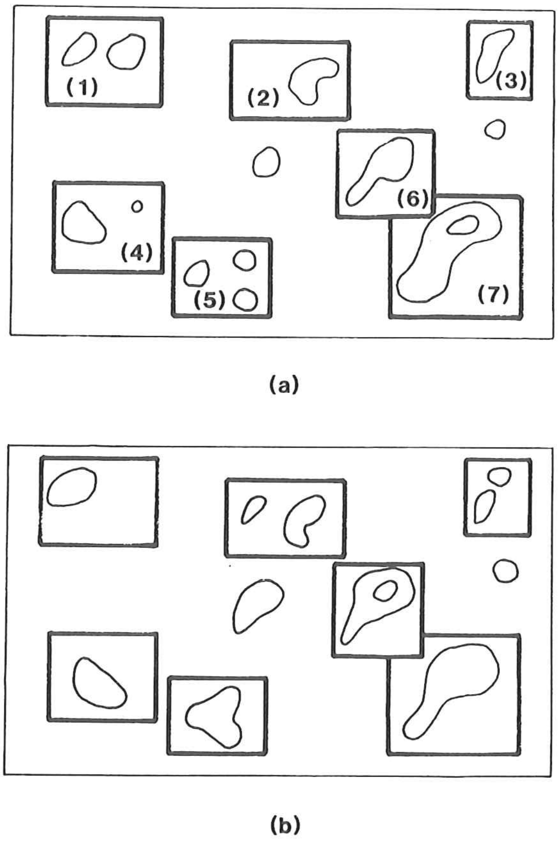


Figure 5. Events that yield contributions to the volume tangent count are inferred from comparisons of (a) and (b).
(1) Two features join to form one: $T_{+-} = 1$.
(2) A new feature appears on plane (b): $T_{++} = 1$.
(3) One feature branches into two: $T_{+-} = 1$.
(4) A feature disappears: $T_{++} = 1$.
(5) Three features combine to form one: $T_{+-} = 2$.
(6) A hole appears in the feature: $T_{--} = 1$.
(7) A hole in the feature disappears: $T_{--} = 1$.

## DISCUSSION

An advantage of carrying out a disector analysis based upon the volume tangent count derives from the fact that tangents occur at points on the surface, and a point lies either inside or outside the volume being sampled. Thus, no corrections for intersections of features with the boundary of the disector are required. Of course, there is some uncertainty about the position of the point of each tangent counted; but that is a resolution problem, determined primarily by the spacing between the sections and not a sample surface bias.

A second advantage of this approach derives from the fact that it applies to structures in which features may be multiply connected. The information supplied is the Euler characteristic, and not the separate values of number and connectivity, which may be of greater interest in many applications. Nonetheless, the result provides unambiguous information. Further, in the two limiting cases of zero connectivity (all features simply connected) and a complete network (one single feature, multiply connected), the volume tangent count yields estimates of number and connectivity, respectively:

$$T_{Vnet} \text{ (simply connected)} = 2N_V \qquad (7a)$$

$$T_{Vnet} \text{ (connected network)} = -2C_V \qquad (7b)$$

In general, for complex structures for which both $N_V$ and $C_V$ have significant values relative to each other, the disector approach will be inadequate to estimate either $N_V$ or $C_V$, and a complete serial sectioning analysis must be undertaken (Aigeltinger et al., 1972).

## SUMMARY

The net volume tangent count provides an unambiguous and unbiased estimate of the net spherical image of a feature set in a real three dimensional microstructure. The net spherical image is in turn related to the Euler characteristic of the feature set. Thus, the volume tangent count provides an unbiased estimate of the Euler characteristic for a microstructure. If the connectivity of the feature set is zero, the tangent count estimates number of features in the set. If the feature set is a single connected network, the tangent count estimates its connectivity.

The net volume tangent count may be performed on a disector, i.e., by comparison of appearances, disappearances and branching events that may be inferred to occur between two closely spaced plane sections through the structure. The resulting estimate is untroubled by sample surface bias effects.

## ACKNOWLEDGMENTS

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