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KRASNOSELSKY-TYPE THEOREMS INVOLVING OUTWARD RAYS [1]

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1. INTRODUCTION.

We know that the kernel of a hunk $S \subseteq \mathbf{R}^n$ can be described as the intersection of the inner stems of its points of local nonconvexity (theorem 4.3 of [4]). Then it is natural to look for Krasnoselsky-type theorems to state the starshapedness of S by means of properties which involve subsets containing finite elements of the set. Helly's theorem needs to be applied to the sets that appear in the characterization, and these sets should be convex. Our problem now is that the inner stems of boundary points are not necessarily convex. To solve this, we prove what is called a "Krasnoselsky-type lemma" which consists in getting a new characterization of the kernel of the set as the intersection of the closures of the convex hull of the inner stems of its points of local nonconvexity. The planar case was solved by F. Toranzos (see [4]). Finally we obtain the Krasnoselsky-type theorems.

More formally, if $S \subseteq \mathbf{R}^n$ hunk then kerS = $\bigcap_{t \in \mathrm{lnc}S} cl(conv(ins(t,S)))$. One of the inclusions is immediate, then we state the problem in the following way:

Let $S \subseteq \mathbf{R}^n$ be a hunk, $x \in S$. If $x \notin$ kerS, then there exists t a point of local nonconvexity of S such that $x \notin$ cl conv (ins(t,S)).

2.- BASIC DEFINITIONS AND NOTATIONS.

Unless otherwise stated all the points and sets considered here are included in $\mathbf{R}^n$ the real n-dimensional euclidean space. The interior, closure, boundary, complement and convex hull of a set S are denoted by: intS, clS, bdryS, CS and convS respectively. The open segment joining x and y is denoted (x,y). The substitution of one or both parentheses by square ones indicates the adjunction of the corresponding extremes.

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We say that *x sees y -via S-* if $[x,y] \subseteq S$. The *star of a point x in S* is the set st(x,S) of all the points of S that see x -via S-, a *star-center* of S is a point $x \in S$ such that st(x,S) = S. The *kernel of S* is the set kerS of all the star-centers of S. S is *starshaped* if kerS $\neq \emptyset$. If S is a connected set with nonempty interior and $x \in S$, then the set of *critical visibility of x* in S is cv(x,S) = intS $\cap$ bdry(st(x,S)). Each point of this set is called *point of critical visibility* of x in S. A point $x \in S$ is called of *local convexity* of S if it admits a neighborhood V such that $V \cap S$ is convex; otherwise it is called of *local nonconvexity*. The set of all the points of local convexity and of local nonconvexity are denoted lcS and lncS respectively. $R(x \rightarrow y)$ is the closed ray issuing from x and going through y, while $R(xy \rightarrow)$ is the closed ray issuing from y and going in the same direction to that of $R(x \rightarrow y)$. Given $y \in$ bdryS and $x \in$ st(y,S) we say that the ray $R(x \rightarrow y)$ is *inward through y* if there exists $t \in R(xy \rightarrow)$ such that $(y,t) \subset$ intS. Otherwise it is called an *outward ray through y*. The *inner stem of y* with respect to S is the set ins(y,S) formed by y and all the points of st(y,S) that issue outward rays through y. A *hunk* is a bounded set S such that intS is connected and S = cl(intS). Let $p, q \in S$, p is said to have *higher visibility than q via S* if $st(q,S) \subseteq st(p,S)$. A *peak* of S is a point $p \in S$ that admits a neighborhood U such that p has higher visibility via S than any other point of $U \cap S$. Given $S \subseteq \mathbf{R}^n$ and $\varepsilon > 0$, we define the *outer* ε *-parallel* and the *inner* ε *-parallel* of S, which we denote S_ε and $S_{-\varepsilon}$ respectively, in the following way $S_\varepsilon = B(S, \varepsilon) = \cup\{ B(x, \varepsilon) \,/\, x \in S \}$ where $B(x, \varepsilon)$ is the closed ball with center in x and radius ε and $S_{-\varepsilon} = B(S,-\varepsilon) = cl(CB(CS, \varepsilon))$. The following lemma has straightforward proof.

Lemma: *$A, M \subseteq R^n$, $\varepsilon > 0$, then $M \subseteq B(A, -\varepsilon)$ if and only if $B(M, \varepsilon) \subseteq A$.*

3.- BASIC CONSTRUCTIONS.

First we will study in this paragraph some detailed properties of a certain curve and a "tube" which will be the main tools in our proof.

The following analysis is based on the hypothesis of our problem, hence we will assume that $S \subseteq \mathbf{R}^n$ is a hunk, $x \in S$ and $x \notin$ kerS. If $x \notin$ kerS there exists some point $q \in S$ such that x does not see q via S, i.e. $x \notin st(q,S)$. The star of any point of S is a closed set, then there exists x' $\in$ intS such that x' $\notin$ st(q,S). Analogously $q \notin st(x,S)$ then there exists q' $\in$

intS such that q' $\notin$ st(x,S). Hence, from the beginning we suppose -without loss of generality- that x and q are interior points. Since intS is connected and open there must exist an arc of simple curve $\Omega \subseteq$ intS that connects x and q. By means of a standard compactness argument we can obtain $\varepsilon > 0$ such that $\Omega_\varepsilon \subseteq$ intS. The previous lemma implies that $\Omega \subseteq S_{-\varepsilon}$ and, a fortiori, x, q $\in S_{-\varepsilon}$ but they do not see each other via $S_{-\varepsilon}$. Let us define the family $\mathcal{F} = \{ \Lambda / \Lambda$ arc of simple curve that connects x and q, $\Lambda \subset S_{-\varepsilon} \}$. It is not an empty family since we have shown $\Omega \in \mathcal{F}$. Hence we can select $\Gamma \in \mathcal{F}$ having minimum length. Following Stavrakas' argument (see [2]) we know that there exist a first point $z_1 \in \mathrm{lnc}(S_{-\varepsilon})$ (going from x towards q) and a first point $z_2 \in \mathrm{lnc}(S_{-\varepsilon})$ (going from q towards x) such that the first and last spans of the curve are line segments $[x, z_1]$ and $[z_2, q]$ respectively.

The geometrical characteristics of $S_{-\varepsilon}$ let us ensure the existence of an interior point q', which is not seen by x, such that Γ admits the configuration $\Gamma = [x, z_1) \cup \overline{\Gamma_1} \cup (z_2, q']$, where $\overline{\Gamma_1}$ is an arc of a non-degenerated curve, different from a line segment. Without loss of generality we suppose that the point q is such a point. Notice that we can choose a priori $\varepsilon > 0$ small enough so that $[x, z_1) \subseteq \mathrm{int}(S_{-\varepsilon})$ and $(z_2, q] \subseteq \mathrm{int}(S_{-\varepsilon})$. Then $\overline{\Gamma_1}$ will be the only arc of Γ wholly included in $\mathrm{bdry}(S_{-\varepsilon})$. (*)

We now study some useful properties of the curve Γ constructed above.

Proposition 3.1: *Consider the curve Γ constructed above, then:*

a) $\overline{\Gamma_1} \subseteq lnc(S_{-\varepsilon})$.

b) Let be $p \in bdry(\Gamma_\varepsilon)$, and $c \in \Gamma$ such that $p \in bdry(B(c, \varepsilon))$

b.1) If $p \in bdryS$, then $p \in lncS$.

b.2) If $p \in lcS$, then $c \in lc(S_{-\varepsilon})$.

c) $(\overline{\Gamma_1})_\varepsilon \cap bdryS \subset lncS$.

Proof. a) Let us take $p \in \overline{\Gamma_1}$ and suppose $p \in \mathrm{lc}(S_{-\varepsilon})$. This means that there exists $U = B(p,\delta)$ a closed neighborhood such that $U \cap S_{-\varepsilon}$ is convex. Let *a* be the entry point of $\overline{\Gamma_1}$ into U and *b* the exit point from U (going from x to q). Since $a, b \in U \cap S_{-\varepsilon}$ which is convex, it results $[a,b] \subset U \cap S_{-\varepsilon} \subset S_{-\varepsilon}$ and $\overline{\Gamma_1}$ would not be of minimum length.

b.1) Suppose that $c \in \text{lnc}(S_{-\varepsilon})$, then there exist c_1 and c_2 close enough to c such that $[c_1, c_2]$ is not included in $S_{-\varepsilon}$. If we pick U any neighborhood of p, let us take the points $p_i \in \text{bdry}(B(c_i, \varepsilon)) \cap \text{bdry}S$ $(i = 1,2)$. It is easy to see that $[p_1, p_2]$ can not be included in S because this would contradict the initial assumption.

b.2) If $p \in \text{bdry}(\Gamma_\varepsilon)$, there exists $c \in \Gamma$ such that $p \in \text{bdry}(B(c,\varepsilon))$ and if $p \in \text{lc}S$, then $c \in \text{lc}(S_{-\varepsilon})$ and $c \in \text{bdry}(S_{-\varepsilon})$, but then due to (*) $c \in \overline{\Gamma_1}$ and using a), $c \in \text{lnc}(S_{-\varepsilon})$ which is absurd. Then $p \in \text{lnc}S$.

c) It is immediate from a) and b).■

From now on, we denote $T = \Gamma_\varepsilon = B([x, z_1), \varepsilon) \cup (\overline{\Gamma_1})_\varepsilon \cup B((z_2, q], \varepsilon)$ which is a "tube" included in S, where $B([x, z_1), \varepsilon)$ and $B((z_2, q], \varepsilon)$ are cylinders wholly included in intS, and $(\overline{\Gamma_1})_\varepsilon \cap \text{bdry}S$ is an arc of curve formed by points of local nonconvexity of S.

4. THE MAIN THEOREM.

Krasnoselsky proved that given a compact and connected set S, $S \subseteq \mathbf{R}^n$, and points $x, y \in S$ such that y does not see x via S, then there exist $z \in S$ and H hyperplane through z which separates x from st(z,S) (see [5]). Since the inner stem of a boundary point is included in its star, the Krasnoselsky separation would be enough for our purposes if we ensure that the point z of contact of the separating hyperplane is a point of local nonconvexity of S.

Theorem 4.1: *$S \subseteq \boldsymbol{R}^n$, $n \geq 2$ a hunk, $x \in S$.*

If $x \notin \text{ker}S$, then there exists $z \in \text{lnc}S$ such that $x \notin \text{cl conv}(\text{ins}(z,S))$.

Proof. We try to find a point $z \in \text{lnc}S$ and an hyperplane H through z such that ins(z,S) is included in H^+, and $x \in H^-$ where H^+ and H^- are the closed and open half-spaces determined by H respectively. Immediately we will have the thesis.

As we did in paragraph 3 we can consider $x, q \in \text{int}S$ such that x does not see q via S and $\Gamma = [x, z_1) \cup \overline{\Gamma_1} \cup (z_2, q]$ the minimum length curve built for a certain $\varepsilon > 0$, and we denote $T = B(\Gamma, \varepsilon)$.

As x does not see q via S, x does not see q via T, hence $x \notin \text{ker}T$, and we know (see [5]) that x is not a peak of T. Then, given any neighborhood U of x, we can always pick some point $x' \in U \cap T$ such that x' sees -via T- some point that x does not see. Let us take $U = B(x,\varepsilon)$ and let $x' \in B(x,\varepsilon) \cap T$ be a point such that st(x',T) is not included in st(x,T); $x' \in$ intS by construction. We pick $p \in \Gamma$ the last point of Γ visible from x (going from x towards q), and $s \in \Gamma$ the last point of Γ visible from x' (going from x' towards q); $p \in$ cv(x,S) and $s \in$ cv(x',S). Then, there exist two points of local nonconvexity of S such that $t \in (x, p) \cap$ lncS and $y \in (x', s) \cap$ lncS (Theorem 2.1 [4]).

Now we consider $c \in \Gamma$ such that $B(c,\varepsilon)$ is the last ball (going from x towards q), entirely seen from x. Notice that $t \in$ bdry$B(c,\varepsilon)$. Analogously we take $d \in \Gamma$ such that $B(d,\varepsilon)$ is the last ball (going from x' towards q), entirely seen from x' and we have $y \in$ bdry$B(d,\varepsilon)$. Notice that by construction $c \neq d$ even though $p = s$ or not.

We consider two cases:

(i) $t = y$, (see fig. 1. In this figure $c_1 = z_1$, $d_1 = z_2$).

x sees y because x sees t, but it does not see $B(d,\varepsilon)$ completely, then the line L(x,y) through x and y is tangent to $B(c,\varepsilon)$ at y but pierces $B(d,\varepsilon)$ at y. Let H be the hyperplane tangent to $B(d,\varepsilon)$ at y. It verifies that L(x,y) is not included in it, hence $x \notin H$. If we call H^+ the closed half-space determined by H in which $B(d,\varepsilon)$ is included, we have $x \in H^-$. Then the z wanted is t.

(ii) $t \neq y$ (see fig.2)

Let be $\Gamma_1 \subset \overline{\Gamma_1}$ the subarc of $\overline{\Gamma_1}$ which joins z_1 and d, and $T' = B([x, z_1) \cup \Gamma_1, \varepsilon)$. By construction x does not see y -via T'- then, applying Krasnoselsky's Lemma to x and y in T' we have that there exist a point z in bdryT' and an hyperplane H_0 through z which separates x from st(z,T'). It is clear that this hyperplane separates x from ins(z,T').

We have $t, y \in$ bdryT' $\cap$ bdryS, then by means of 3.1 (b.1) the arc $\Gamma_2 \subset \overline{\Gamma_1}$ that connects c and d verifies $\Gamma_2 \subset$ lnc($S_{-\varepsilon}$) and then $B(\Gamma_2, \varepsilon) \cap$ bdryS $\subset$ lncS.

When we use the Krasnoselsky's lemma for x and y, that means "pushing" (in the direction from x to y) a little enough ball external to T', we get some point z which lies in bdryT'$\cap$ bdryS which is -due to 3.1- a point of local nonconvexity of S.

Then we have found H_0 hyperplane that separates x from ins(z,T'), (the closed and open half-spaces determined by H_0 are denoted H_0^+ and H_0^- respectively). In this case $x \in H_0^-$, ins(z,T') $\subseteq H_0^+$ and $z \in$ lncS. Finally, we prove that H_0 separates x from ins(z,S). If we pick $u \in$ ins(z,S) notice that:

a) if u $\in$ T' then u $\in$ H_0^+.

b) if u $\notin$ T' we consider v $\in$ R(u $\rightarrow$ z) $\cap$ bdryT' (the first point from u towards z) and, as R(u $\rightarrow$ z) is an outward ray through z, R(v $\rightarrow$ z) is also an outward ray through z, then v $\in$ ins(z,T') and v $\in$ H_0^+, then u $\in$ H_0^+. ■

Corollary 4.2: *$S \subseteq R^n$, $n \geq 2$ a hunk, then kerS = $\cap$ { cl(conv(ins(t,S))) / t $\in$ lncS }*

Proof. $\subseteq$) it follows immediately from the characterization of the kernel of S as the intersection of the inner stems of its points of local nonconvexity (theorem 4.3 of [4]).

$\supseteq$) it is theorem 4.1. ■

5.- KRASNOSELSKY-TYPE THEOREMS.

Theorem 5.1: *Let $S \subseteq R^n$ be a nonconvex hunk such that for every k-pointed set M $\subseteq$ lncS (with $k \leq n$) there exists a point p $\in$ S that sees each of the points of M and issues outward rays through these points. Then S is starshaped.*

Proof. It is immediate applying Helly's Theorem (see [3]) to the following family: $\mathcal{F}$ = { cl conv (ins(y,S)) / y $\in$ lncS } and using corollary 4.2. ■

Theorem 5.2: *Let $S \subseteq R^n$ be a nonconvex hunk and $\delta > 0$ such that for every k-pointed set M $\subseteq$ lncS (with $k \leq n + 1$) there exists a disk D of radius δ included in the star of each point of M, and such that every point of D issues an outward ray through each point of M. Then kerS includes a disk of radius δ.*

Proof. It is immediate using Klee's Theorem (see [3]) and Corollary 4.2. ■

Theorem 5.3: *Let $S \subseteq R^n$ be a nonconvex hunk such that lncS be finite and for every k-pointed set M $\subseteq$ lncS (with $k < n + 1$) there exists a segment I included in the star of each point of M, and such that every point of I issues an outward ray through each point of M. Then kerS has dimension at least 1.*

This theorem only has sense in dimension 2, case proved by F. Toranzos in [4], because M. Breen showed in [1] that if lncS is a finite set, then S is a planar set.

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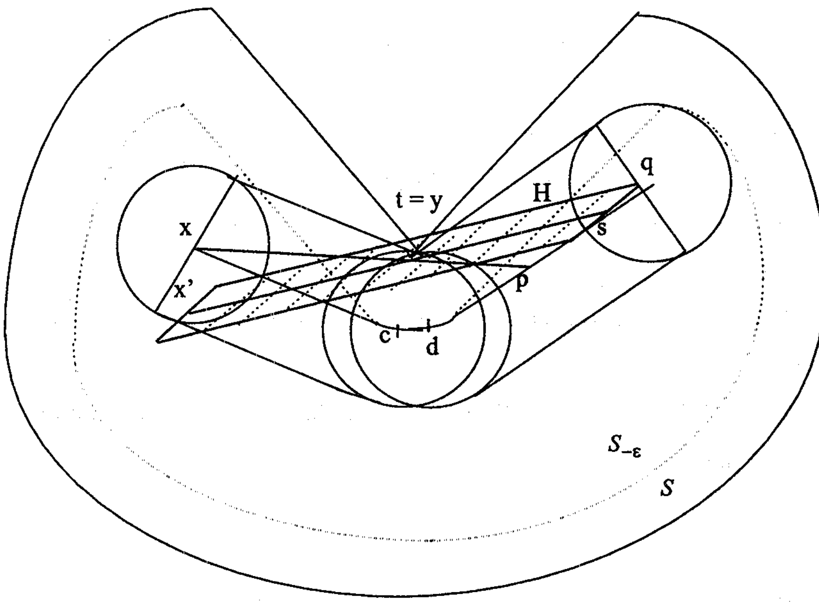
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Fig. 1

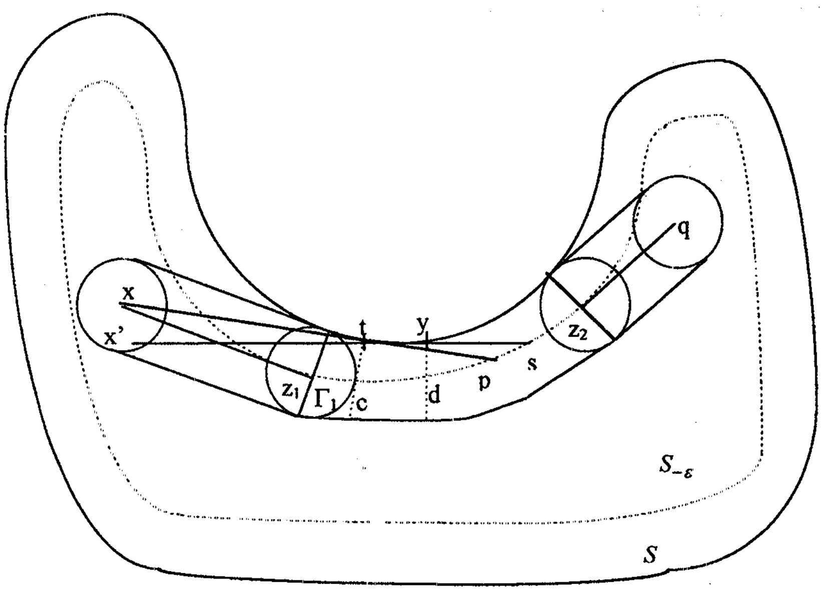
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Fig. 2